Bidual octahedral renormings and strong regularity in Banach spaces

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Outline of the talk

The talk is based on the paper:
J. L. and G. López-Pérez, *Bidual octahedral renormings and strong regularity in Banach spaces*, J. Inst. Math. Jussieu (2019), 1–17.

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- Background
- Octahedral norms
- The problem
- Connection with diameter 2 properties
- Results

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Remark (R. Haydon, 1982)

Maurey's theorem is false for nonseparable spaces.

Octahedral norms

Definition (G. Godefroy, B. Maurey, 1989)

Let X be an infinite-dimensional Banach space. The norm on X is octahedral (OH) if, for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there is a $y \in S_X$ such that

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ℓ_1, $C[0,1]$, $L_1[0,1]$, and $L_\infty[0,1]$ are OH.

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$$|||x + x^{**}||| = |||x||| + |||x^{**}||| \quad \textit{for all } x \in X.$$

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Problem (G. Godefroy, 1989)

If a Banach space X contains an isomorphic copy of ℓ_1, does there always exist an equivalent norm on X such that the bidual X^{**} is OH?

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If a Banach space X contains an isomorphic copy of ℓ_1, does there always exist an equivalent norm on X such that the bidual X^{**} is OH?

Partial answer: Yes, if X is separable.

Diameter 2 properties

Recall that a slice of B_X is a set of the form

$$S(B_X, x^*, \alpha) := \{x \in B_X \colon x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

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Definition (T. A. Abrahamsen, V. Lima, O. Nygaard, 2013)

A Banach space X has the strong diameter 2 property (SD2P) if every average of slices of B_X has diameter 2, that is,

$$\forall n \in \mathbb{N}, \quad \operatorname{diam}\left(\frac{1}{n}\sum_{i=1}^{n} S_i(B_X, x_i^*, \alpha_i)\right) = 2.$$

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c_0, ℓ_∞, $C[0,1]$, $L_1[0,1]$, and $L_\infty[0,1]$ all have the SD2P.

Duality between SD2P and OH

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A Banach space is said to be strongly regular if every closed, bounded and convex subset of X contains convex combinations of slices with diameter arbitrarily small.

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It is known that (see, e.g., GGMS):

- X^* is strongly regular if and only if $X \not\supseteq \ell_1$.

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Hence, Godefroy's problem is a particular (dual) case of the following:

A bigger problem (J. Becerra Guerrero, G. López-Pérez, and A. Rueda Zoca, 2015)

Can every Banach space failing to be strongly regular be equivalently renormed such that it has the strong diameter two property?

Talagrand's example revisited

Theorem (M. Talagrand, 1989)

There exists a convex weak* compact subset $\mathcal{T}$ of $S_{C(\Delta)^*}$, where $\Delta = \{0,1\}^{\mathbb{N}}$, such that every average of slices of $\mathcal{T}$ has diameter 2.

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Fix $s \in \{3,4,\dots\}$. For $I \subset \mathbb{N}$, $i \in \mathbb{N}$, define on the ith copy of $\{0,1\}$ a measure

$$\nu_{s,I}^{(i)} := \begin{cases} \frac{1}{s}\delta_0^{(i)} + \frac{s-1}{s}\delta_1^{(i)} & \text{if } i \in I \\ \frac{s-1}{s}\delta_0^{(i)} + \frac{1}{s}\delta_1^{(i)} & \text{if } i \notin I. \end{cases}$$

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Now let $(N_s)_{s\geq 3}$ be a partition of $\mathbb{N}$ into disjoint infinite sets and define a probability measure on Δ by $\rho_I := \bigotimes_{s\geq 3} \bigotimes_{i\in N_s} \nu_{s,I}^{(i)}$.

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Consider the operator $T\colon C(\Delta) \to C(\Delta)$ defined by $T(f)(I) := \rho_I(f)$, for every $f \in C(\Delta)$, $I \subset \mathbb{N}$. Then $T^*(\delta_I) = \rho_I$, where δ_I is the Dirac measure at I on Δ.

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Finally, let $\mathcal{T} := T^*(\mathbb{P}_\Delta)$, where $\mathbb{P}_\Delta$ is the set of probability measures on Δ.

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Denote by $\mathcal{Y} := \operatorname{span} \mathcal{T}$ and $\mathcal{K} := \operatorname{conv}(\mathcal{T} \cup -\mathcal{T})$.

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4. *there exists a sequence* $(f_k)_{k\in\mathbb{N}} \subset C(\Delta)$ *such that* $\operatorname{span}\{f_k\} \simeq \ell_1$ *in* $C(\Delta)$ *and* $\operatorname{span}\{f_{k|\mathcal{Y}}\} \simeq \ell_1$ *in* $\mathcal{Y}_*$;

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5. $(\mathcal{Y}_*, |\cdot|_*)^{**}$ *is OH. In particular, for every* $n \in \mathbb{N}$, $y_1^{**}, \ldots, y_n^{**} \in B_{(\mathcal{Y}_*,|\cdot|_*)^{**}}$, *and* $\varepsilon > 0$ *there is a* $k \in \mathbb{N}$ *such that*

$$|y_i^{**} + f_{k|\mathcal{Y}}|^* \geq (1-\varepsilon)(|y_i^{**}|^* + 1) \quad \text{for every } i \in \{1, \ldots, n\}.$$

Abstract renorming

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(A2) *there exists Z a subspace of X such that $S|_Z : (Z, \|\cdot\|_X) \to (\overline{\text{span}}(A), \|\cdot\|_Y)$ is an onto isomorphism.*

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Then there is an equivalent norm in X such that the bidual X^{**} is OH.

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Sketch of the proof.

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- Let $j_{\mathcal{Y}}$ the natural bounded linear map from $C(\Delta)$ to $C(\Delta)|_{\mathcal{Y}}$.
- Let i be the inclusion map from $C(\Delta)|_{\mathcal{Y}}$ to $\mathcal{Y}_*$.

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Let X be a separable Banach space. TFAE:

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A word about the non-separable case

Corollary (J. L., G. López-Pérez)

Let X be Banach space. If there exists a closed subspace Y such that X/Y is separable and contains a subspace isomorphic to ℓ_1, then there is an equivalent norm in X such that the bidual X^{**} is OH.

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Does there exist an equivalent norm in ℓ_∞ such that its bidual ℓ_∞^{**} is OH?

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