# Banach spaces where convex combinations of slices are relatively weakly open.

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Joint work with Trond Abrahamsen, Julio Becerra Guerrero, Rainis Haller and Märt Põldvere

$X$ Banach space with unit ball $B_X$, unit sphere $S_X$ and dual $X^*$.
For $0 < \alpha < 2$, $x^* \in S_{X^*}$, a *slice* of $B_X$ is a set

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## Lemma [Bourgain]

*Simplified version:* For any non-empty relatively weakly open subset $U$ of $B_X$, there exists positive scalars $\lambda_1, \ldots, \lambda_p$ with $\sum_{i=1}^{p} \lambda_i = 1$ and slices $S_1, \ldots, S_p$ of $B_X$ so that $\sum_{i=1}^{p} \lambda_i S_i \subset U$.

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That is: Relatively weakly open subsets of the unit ball contain convex combinations of slices. We will look at a converse to this.

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Does there exist Banach spaces $X$ for which every (finite non-empty) convex combination of slices is relatively weakly open in $B_X$?

We say that $X$ **is CWO** if it has this property.

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**Theorem [López-Pérez, Martín, and Rueda Zoca]**

If $X$ is infinite dimensional and CWO then every convex combination of slices of $B_X$ has diameter 2.
In particular, $X^*$ is octahedral.

# Property (co)

### Definition

$x \in B_X$ has *property (co)*, if for every $n \geq 2$,

- whenever $\varepsilon > 0$ and $x = \sum_{j=1}^{n} \lambda_j x_j$, where $x_1, \dots, x_n \in B_X$ and $\lambda_1, \dots, \lambda_n > 0$, $\sum_{j=1}^{n} \lambda_j = 1$,

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We say that the space $X$ has *property (co)* if every point $x \in B_X$ has property (co).

### Theorem [Abrahamsen, Becerra Guerrero, Haller, L., and Põldvere]

If $K$ is scattered compact Hausdorff, and $X$ is finite dimensional with property (co), then

- $C(K, X)$ is CWO.
- $X$ is CWO.

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### Theorem [Kadets]

If $X$ is finite dimensional and CWO, then $X$ has property (co).

The proof that $C(K, X)$ is CWO whenever $K$ is scattered and $X$ is finite dimensional with property (co) is a bit technical.
Next is a proof that $c_0$ is CWO to get a taste of the main ideas.

*Proof:* Consider $C := \sum_{i=1}^{n} \lambda_i S(f_i, \alpha_i)$, where $\lambda_i > 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, $f_i \in \ell_1$ with $\|f_i\| = 1$, and $\alpha_i > 0$.

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For $z = \sum_{i=1}^{n} \lambda_i x_i \in C$ pick $\varepsilon > 0$ such that

$$\langle x_i, f_i \rangle > 1 - \alpha_i + \varepsilon.$$

Find a finite set $F$ such that $\sum_{t \notin F} |f_i(t)| < \varepsilon/3$.

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By property (co) of $\mathbb{R}$ there exists a $\delta > 0$ and continuous functions $\phi_{t,i} : B(z(t), \delta) \cap B_{\mathbb{R}} \to B_{\mathbb{R}}$ such that $u = \sum_{i=1}^n \lambda_i \phi_{t,i}(u)$ and $|\phi_{t,i}(u) - x_i(t)| < \varepsilon/3$ for all $u \in B(z(t), \delta) \cap B_{\mathbb{R}}$.

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Define

$$\mathcal{U} := \{y \in B_{c_0} : |y(t) - z(t)| < \delta, t \in F\}.$$

For $y \in \mathcal{U}$ define

$$y_i(t) := \begin{cases} y(t) & t \notin F, \\ \phi_{t,i}(y) & t \in F. \end{cases}$$

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Then $y_i \in B_{c_0}$, $y = \sum_{i=1}^n \lambda_i y_i$, and

$$\langle y_i, f_i \rangle \geq \sum_{t \in F} f_i(t) x_i(t) + \sum_{t \in F} f_i(t)(y_i(t) - x_i(t)) - \frac{\varepsilon}{3} > 1 - \alpha_i.$$