On octahedrality of Müntz spaces

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Müntz spaces

Let $\Lambda = (\lambda_i)_{i=0}^{\infty}$, with $\lambda_0 = 0$, be a strictly increasing sequence of non-negative real numbers. We will call $M(\Lambda) := \overline{\text{span}}\{t^{\lambda_i}\}_{i=0}^{\infty} \subseteq C[0,1]$ a Müntz space if $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} < \infty$.

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Theorem (Müntz 1914)

The space $M(\Lambda)$ is equal to $C[0,1]$ if and only if

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty.$$

Content of the presentation

i. Müntz spaces embed isomorphically into c_0,
ii. No Müntz space is locally octahedral,
iii. No Müntz space is almost square.

Theorem (Bounded Bernstein's inequality)

Assume that $1 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ *and* $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$, *then for every* $\varepsilon > 0$ *there is a constant* K_ε *such that*

$$\|p'\|_{[0,1-\varepsilon]} \leq K_\varepsilon \|p\|_{[0,1]},$$

for all $p \in span\{t^{\lambda_i}\}_{i=0}^{\infty}$.

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Sketch of the construction of J_ε: Let $\varepsilon > 0$ and choose a sequence $0 = a_0 < a_1 < \cdots < a_i < \cdots < 1$ converging to 1. Pick points $0 = s_0 < s_1 < \cdots < s_{n_1} = a_1 < s_{n_1+1} < \cdots < s_{n_2} = a_2 < \cdots$, in such a way that

$$s_{j+1} - s_j \leq \frac{\varepsilon}{K_{i+1}} \text{ for } n_i \leq j < n_{i+1}.$$

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Define $J_\varepsilon : M(\Lambda) \to c$ by $J_\varepsilon(f) = (f(s_n))_n$

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Note that this implies that $M(\Lambda)^*$ is separable.

Definition

Let X be a Banach space. Then X is

(i) *locally octahedral* (LOH) if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.

(ii) *octahedral* (OH) if for every $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm y\| > 2 - \varepsilon$ for all $i \in \{1, \ldots, n\}$.

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Proof: There exists w^*-slices $S(x, \varepsilon)$ of the unit ball of $M(\Lambda)^*$ of arbitrarily small diameter.

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No Müntz space $M(\Lambda)$ is LOH.

Proof: There exists w^*-slices $S(x, \varepsilon)$ of the unit ball of $M(\Lambda)^*$ of arbitrarily small diameter. This is equivalent to $M(\Lambda)$ failing to be LOH.

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Let X be a Banach space. Then X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $(y_n)_{n=1}^{\infty}$ in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.

(ii) *almost square* (ASQ) if for every $x_1, \ldots, x_k \in S_X$ there exists a sequence $(y_n)_{n=1}^{\infty}$ in B_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \ldots, k\}$.

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No Müntz space $M(\Lambda)$ is LASQ. Just consider the constant function 1. Consider the subspace $M_0(\Lambda) := \overline{\text{span}}(t^{\lambda_n})_{n=1}^{\infty}$ of $M(\Lambda)$.

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No Müntz space $M(\Lambda)$ is LASQ. Just consider the constant function 1. Consider the subspace $M_0(\Lambda) := \overline{\text{span}}(t^{\lambda_n})_{n=1}^{\infty}$ of $M(\Lambda)$.

No Müntz space $M_0(\Lambda)$ with $\lambda_1 \geq 1$ is LASQ.

Every Müntz space $M_0(\Lambda)$ have a subspace of finite codimension which is not ASQ. This is equivalent to $M_0(\Lambda)$ not being ASQ.

Thanks you for your attention!

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