Subseries convergence and the possibility of laziness

Subseries convergence and the possibility of laziness

“Very old” work (2001)

Subseries convergence and the possibility of laziness

“Very old” work (2001)
partly joint with A. Borichev and R. Deville

Smooth functions on c_0

.

Smooth functions on c_0

Theorem (Hájek 1998).

.

Smooth functions on c_0

Theorem (Hájek 1998). *If* $f : c_0 \to \mathbb{R}$ *is* $\mathcal{C}^1$*-smooth*

.

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0},*

.

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} .*

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property:

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big)$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence

$$\inf_{1\le i\le n} |f(x+h_i)-f(x)|$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence

$$\inf_{1\leq i\leq n} |f(x+h_i)-f(x)| \xrightarrow{n\to\infty} 0 \qquad .$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence

$$\inf_{1\le i\le n} |f(x+h_i)-f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B_{c_0}.$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence (f real-valued)

$$\inf_{1\le i\le n} |f(x+h_i)-f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B_{c_0}.$$

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence (f real-valued)

$$\inf_{1\le i\le n} |f(x+h_i)-f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B_{c_0}.$$

(ii) This implies

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence (f real-valued)

$$\inf_{1\le i\le n} |f(x+h_i)-f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B_{c_0}.$$

(ii) This implies that f *turns w-Cauchy sequences in B_{c_0} into Cauchy sequences in $\mathbb{R}$.*

Smooth functions on c_0

Theorem (Hájek 1998). *If $f : c_0 \to \mathbb{R}$ is $\mathcal{C}^1$-smooth and such that f' is uniformly continuous on B_{c_0}, then f is uniformly continuous on B_{c_0} with respect to the weak topology.*

`One possible proof.`

(i) f has the following property: for any sequence $(h_i) \xrightarrow{w} 0$ in c_0,

$$\frac{1}{|I|}\sum_{i\in I}\big(f(x+h_i)-f(x)\big) \xrightarrow{|I|\to\infty} 0 \qquad \text{uniformly on } B_{c_0};$$

and hence (f real-valued)

$$\inf_{1\le i\le n} |f(x+h_i)-f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B_{c_0}.$$

(ii) This implies that f *turns w-Cauchy sequences in B_{c_0} into Cauchy sequences in $\mathbb{R}$.*

(iii) OK since any bounded sequence in c_0 has a w-Cauchy subsequence.

Strongly sequentially continuous functions

Strongly sequentially continuous functions

Definition.

Strongly sequentially continuous functions

Definition. G topological abelian group,

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$.

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if,

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Equivalently:

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \leq i \leq n} |f(x+h_i) - f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

$$\lim_{n \to \infty} \inf |f(x_n + h_i) - f(x_n)|$$

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \leq i \leq n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

$$\liminf_{n \to \infty} |f(x_n + h_i) - f(x_n)| \xrightarrow{i \to \infty} 0.$$

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x + h_i) - f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

$$\liminf_{n\to\infty} |f(x_n + h_i) - f(x_n)| \xrightarrow{i\to\infty} 0.$$

- SSC functions turn Cauchy sequences into Cauchy sequences.

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x+h_i) - f(x)| \xrightarrow{n\to\infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

$$\liminf_{n\to\infty} |f(x_n+h_i) - f(x_n)| \xrightarrow{i\to\infty} 0.$$

• SSC functions turn Cauchy sequences into Cauchy sequences.

SSC

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \leq i \leq n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

$$\liminf_{n \to \infty} |f(x_n + h_i) - f(x_n)| \xrightarrow{i \to \infty} 0.$$

- SSC functions turn Cauchy sequences into Cauchy sequences.

$$\text{SSC} \implies \text{sequentially continuous.}$$

Strongly sequentially continuous functions

Definition. G topological abelian group, $B \subseteq G$. A function $f : G \to \mathbb{R}$ is *strongly sequentially continuous* (SSC) on B if, for any sequence $(h_i) \to 0$ in G,

$$\inf_{1 \le i \le n} |f(x + h_i) - f(x)| \xrightarrow{n \to \infty} 0 \qquad \text{uniformly on } B.$$

Equivalently: for any sequence $(x_n) \subseteq B$ and any $(h_i) \to 0$ in G,

$$\liminf_{n \to \infty} |f(x_n + h_i) - f(x_n)| \xrightarrow{i \to \infty} 0.$$

- SSC functions turn Cauchy sequences into Cauchy sequences.
- *Uniformly* SC on $G \implies$ SSC $\implies$ sequentially continuous.

Question.

Question. Does "SSC on G" imply "USC on G"?

Question. Does “SSC on G” imply “USC on G”?

Example 1.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.`

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x) \sin(k!\pi x)$ when $x \in [k, k+1]$,

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x) \sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness:

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q} \;:\; k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}\ :\ k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \le 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.`

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q} \ : k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q} \;:\; k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

$(x_n), (y_n) \subseteq \mathbb{R}$ such that $y_n - x_n \to 0$ and $|f(y_n) - f(x_n)| > \varepsilon > 0$.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

$(x_n), (y_n) \subseteq \mathbb{R}$ such that $y_n - x_n \to 0$ and $|f(y_n) - f(x_n)| > \varepsilon > 0$.
WLOG $|y_n - x_n| \leq 2^{-n}$.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

$(x_n), (y_n) \subseteq \mathbb{R}$ such that $y_n - x_n \to 0$ and $|f(y_n) - f(x_n)| > \varepsilon > 0$.
WLOG $|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

$(x_n), (y_n) \subseteq \mathbb{R}$ such that $y_n - x_n \to 0$ and $|f(y_n) - f(x_n)| > \varepsilon > 0$.
WLOG $|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q} \ : k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow \limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi\,|h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

$(x_n), (y_n) \subseteq \mathbb{R}$ such that $y_n - x_n \to 0$ and $|f(y_n) - f(x_n)| > \varepsilon > 0$.
WLOG $|y_n - x_n| \leq 2^{-n}$.

$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.

Triangle inequality $\implies$

Question. Does "SSC on G" imply "USC on G"?

Example 1. No if $G := \mathbb{Q}$.

`Sketch proof.` Consider the even function $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = \sin(\pi x)\sin(k!\pi x)$ when $x \in [k, k+1]$, $k \in \mathbb{Z}_+$. (SSC-ness: $\forall h \in \mathbb{Q}$: $k!\pi h \in 2\pi\mathbb{Z}$ if k is large enough $\rightsquigarrow$ $\limsup_{|x|\to\infty} |f(x+h) - f(x)| \leq 2\pi |h|$.)

Example 2. Yes if $G := \mathbb{R}$.

`Sketch proof.` $f : \mathbb{R} \to \mathbb{R}$ not UC. Goal: f not SSC.

$(x_n), (y_n) \subseteq \mathbb{R}$ such that $y_n - x_n \to 0$ and $|f(y_n) - f(x_n)| > \varepsilon > 0$.
WLOG $|y_n - x_n| \leq 2^{-n}$.

$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.

Triangle inequality $\implies$ $\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.

Lemma.

Lemma. *G topological abelian group,*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent $(\text{i.e. } \sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}})$.*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G.*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent (i.e. $\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$.*

Lemma. *G topological abelian group,* $(g_n)_{n\in\mathbb{N}} \subseteq G$ *such that the series* $\sum g_n$ *is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let* A_n, B_n *be Borel subsets of* G. *Assume that* $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. *Then one can find* $h \in G$

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \;:\; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$)*. Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. *$\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. *$\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$*). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent (i.e. $\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent $(\text{i.e. } \sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}})$. Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \;:\; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent (i.e. $\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \;:\; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)$

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent $(\text{i.e. } \sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}})$. Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \;:\; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ + diagonal argument

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent (i.e. $\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ + diagonal argument $\Longrightarrow$

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ **+** diagonal argument $\implies \ \exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \;:\; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ **+** diagonal argument $\implies$ $\exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i \;:\; h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$,

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N}\ :\ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ **+** diagonal argument $\implies\ \exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i\ :\ h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$, or (b) same with B_n.

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ + diagonal argument $\implies$ $\exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i \ : \ h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$, or (b) same with B_n. WLOG (a)

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \ : \ G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n \geq i}$ for each $i \in \mathbb{N}$ **+** diagonal argument $\implies \ \exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i \ : \ h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$, or (b) same with B_n. WLOG (a) and $\mathbf{N} = \mathbb{N}$.

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \,:\, G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ + diagonal argument $\implies \exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i \,:\, h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$, or (b) same with B_n. WLOG (a) and $\mathbf{N} = \mathbb{N}$. Then

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent (i.e. $\sum \theta_n g_n$ is convergent for any $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \; : \; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n\geq i}$ for each $i \in \mathbb{N}$ + diagonal argument $\implies \exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i \; : \; h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$, or (b) same with B_n. WLOG (a) and $\mathbf{N} = \mathbb{N}$. Then $\forall i \; : \; \liminf_{n\to\infty} |f(x_n + h_i) - f(x_n)| \geq \varepsilon/2$.

Lemma. *G topological abelian group, $(g_n)_{n\in\mathbb{N}} \subseteq G$ such that the series $\sum g_n$ is subseries convergent* (i.e. $\sum \theta_n g_n$ *is convergent for any* $(\theta_n) \in \{0,1\}^{\mathbb{N}}$). *Let A_n, B_n be Borel subsets of G. Assume that $\forall n \in \mathbb{N} \; : \; G = A_n \cup (B_n + g_n)$. Then one can find $h \in G$ of the form $h = \sum_{n=1}^{\infty} \theta_n g_n$ such that h belongs to infinitely many A_n or to infinitely many B_n.*

Back to $f : \mathbb{R} \to \mathbb{R}$ not UC and *continuous* (WLOG).

$|y_n - x_n| \leq 2^{-n}$.
$A_n := \{h \in \mathbb{R};\ |f(x_n + h) - f(x_n)| > \varepsilon/2\}$,
$B_n := \{h \in \mathbb{R};\ |f(y_n + h) - f(y_n)| > \varepsilon/2\}$.
$\mathbb{R} = A_n \cup (B_n + y_n - x_n)$.
A_n, B_n *open* in $\mathbb{R}$.

Lemma applied with $(y_n - x_n)_{n \geq i}$ for each $i \in \mathbb{N}$ + diagonal argument $\implies \exists (h_i) \to 0$ in $\mathbb{R}$ and an infinite set $\mathbf{N} \subseteq \mathbb{N}$ such that either (a) $\forall i \; : \; h_i \in A_n$ for all but finitely many $n \in \mathbf{N}$, or (b) same with B_n. WLOG (a) and $\mathbf{N} = \mathbb{N}$. Then $\forall i \; : \; \liminf_{n\to\infty} |f(x_n + h_i) - f(x_n)| \geq \varepsilon/2$.

So f is not SSC.

Proof of the Lemma

Proof of the Lemma

$\sum g_n$ subseries convergent in G

Proof of the Lemma

$\sum g_n$ subseries convergent in G

$G = A_n \cup (B_n + g_n)$ for all $n \in \mathbb{N}$

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted:

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$

Proof of the Lemma

$\sum g_n$ subseries convergent in G

$G = A_n \cup (B_n + g_n)$ for all $n \in \mathbb{N}$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$

Proof of the Lemma

$\sum g_n$ subseries convergent in G

$G = A_n \cup (B_n + g_n)$ for all $n \in \mathbb{N}$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group*

Proof of the Lemma

$\sum g_n$ subseries convergent in G

$G = A_n \cup (B_n + g_n)$ for all $n \in \mathbb{N}$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*),

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.

$J : \Theta \to G$ is continuous

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence),

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$,

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.

$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1)$

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \Longrightarrow$

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.
So $\quad m(\{\theta \in \Theta;\ \theta_n = 1\}) \leq m(J^{-1}(A_n) \cup (J^{-1}(B_n) + \delta^{(n)}))$,

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.
So $1/2 = m\big(\{\theta \in \Theta;\ \theta_n = 1\}\big) \leq m\big(J^{-1}(A_n) \cup (J^{-1}(B_n) + \delta^{(n)})\big)$,

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.
So $1/2 = m\big(\{\theta \in \Theta;\ \theta_n = 1\}\big) \leq m\big(J^{-1}(A_n) \cup (J^{-1}(B_n) + \delta^{(n)})\big)$, and hence

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.
So $1/2 = m\big(\{\theta \in \Theta;\ \theta_n = 1\}\big) \leq m\big(J^{-1}(A_n) \cup (J^{-1}(B_n) + \delta^{(n)})\big)$, and hence $m\big(J^{-1}(A_n)\big) + m\big(J^{-1}(B_n)\big) \geq 1/2$

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.
$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.
So $1/2 = m(\{\theta \in \Theta;\ \theta_n = 1\}) \leq m(J^{-1}(A_n) \cup (J^{-1}(B_n) + \delta^{(n)}))$, and hence $m(J^{-1}(A_n)) + m(J^{-1}(B_n)) \geq 1/2$ for all $n \in \mathbb{N}$

Proof of the Lemma

$$\sum g_n \text{ subseries convergent in } G$$

$$G = A_n \cup (B_n + g_n) \text{ for all } n \in \mathbb{N}$$

Wanted: $\theta = (\theta_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ such that $J(\theta) := \sum_{n=1}^{\infty} \theta_n g_n$ belongs to infinitely many A_n or to infinitely many B_n.

$\Theta := \{0,1\}^{\mathbb{N}}$ Cantor *group* (addition *mod 1*), m Haar measure on Θ.

$J : \Theta \to G$ is continuous (uniform convergence), $J^{-1}(A_n \cup (B_n + g_n)) = \Theta$, and $(\theta_n = 1) \implies J(\theta) = J(\theta - \delta^{(n)}) + g_n$.

So $1/2 = m(\{\theta \in \Theta;\ \theta_n = 1\}) \leq m(J^{-1}(A_n) \cup (J^{-1}(B_n) + \delta^{(n)}))$, and hence $m(J^{-1}(A_n)) + m(J^{-1}(B_n)) \geq 1/2$ for all $n \in \mathbb{N}$ $\rightsquigarrow$ OK.

Adding the triangle inequality

Adding the triangle inequality

Hypolemma.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups,*

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms,*

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$.*

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$,*

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent.*

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

`Proof.`

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \quad : \quad \Phi_n(g_n) \notin W - W$.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \quad : \quad \Phi_n(g_n) \notin W - W$. Define $A_n := \{x \in G;\ \Phi_n(x) \notin W\}$.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \quad : \quad \Phi_n(g_n) \notin W - W$. Define $A_n := \{x \in G;\ \Phi_n(x) \notin W\}$. These A_n are Borel,

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \; : \; \Phi_n(g_n) \notin W - W$. Define $A_n := \{x \in G; \; \Phi_n(x) \notin W\}$. These A_n are Borel, and $G = A_n \cup (A_n - g_n)$ for all $n \in \mathbb{N}$.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

`Proof.` Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \ : \ \Phi_n(g_n) \notin W - W$. Define $A_n := \{x \in G;\ \Phi_n(x) \notin W\}$. These A_n are Borel, and $G = A_n \cup (A_n - g_n)$ for all $n \in \mathbb{N}$. Lemma $\rightsquigarrow$ $x \in G$ such that $\Phi_n(x) \not\to 0$.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \ : \ \Phi_n(g_n) \notin W - W$. Define $A_n := \{x \in G;\ \Phi_n(x) \notin W\}$. These A_n are Borel, and $G = A_n \cup (A_n - g_n)$ for all $n \in \mathbb{N}$. Lemma $\rightsquigarrow$ $x \in G$ such that $\Phi_n(x) \not\to 0$. Contradiction.

Adding the triangle inequality

Hypolemma. *G, H topological abelian groups, $\Phi_n : G \to H$ Borel homomorphisms, $(g_n) \subseteq G$. Assume that $\Phi_n(x) \to 0$ for all $x \in G$, and that $\sum g_n$ is subseries convergent. Then $\Phi_n(g_n) \to 0$.*

Proof. Assume that $\Phi_n(g_n) \not\to 0$. WLOG there exists $W \subseteq H$ open neighbourhood of 0 such that $\forall n \in \mathbb{N} \; : \; \Phi_n(g_n) \notin W - W$. Define $A_n := \{x \in G;\ \Phi_n(x) \notin W\}$. These A_n are Borel, and $G = A_n \cup (A_n - g_n)$ for all $n \in \mathbb{N}$. Lemma $\rightsquigarrow$ $x \in G$ such that $\Phi_n(x) \not\to 0$. Contradiction.

This is unexpectedly useful!

Example 1.

Example 1. Uniform Boundedness Principle without Baire category.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i \in I}$ family of continuous linear operators,

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i \in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space,

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i \in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i \in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i \in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$. (*Lazy construction of a series.*)

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$. (*Lazy construction of a series.*)

$\sum x_n$ is subseries convergent in X.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$. (*Lazy construction of a series.*)

$\sum x_n$ is subseries convergent in X.

$\Phi_n(x) := \frac{1}{n} T_{i_n}(x)$

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$. (*Lazy construction of a series.*)

$\sum x_n$ is subseries convergent in X.

$\Phi_n(x) := \frac{1}{n} T_{i_n}(x) \to 0$ for all $x \in X$.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$. (*Lazy construction of a series.*)

$\sum x_n$ is subseries convergent in X.

$\Phi_n(x) := \frac{1}{n} T_{i_n}(x) \to 0$ for all $x \in X$.

$\Phi_n(x_n) \not\to 0$.

Example 1. Uniform Boundedness Principle without Baire category.

$(T_i)_{i\in I}$ family of continuous linear operators, $T_i : X \to Y$ with X Banach and Y normed space, such that $\sup_{i\in I} \|T_i(x)\| < \infty$ for all $x \in X$.

Assume that $\sup_{i\in I} \|T_i\| = \infty$.

$(x_n) \subseteq X$ and $(i_n) \subseteq I$ such that $\|x_n\| \leq 2^{-n}$ and $\|T_{i_n}(x_n)\| \geq n$. (*Lazy construction of a series.*)

$\sum x_n$ is subseries convergent in X.

$\Phi_n(x) := \frac{1}{n} T_{i_n}(x) \to 0$ for all $x \in X$.

$\Phi_n(x_n) \not\to 0$.

Contradiction.

Example 2.

Example 2. Automatic continuity.

Example 2. Automatic continuity.
X Banach, Y normed space,

Example 2. Automatic continuity.
X Banach, Y normed space, $T : X \to Y$ linear

Example 2. Automatic continuity.
X Banach, Y normed space, $T : X \to Y$ linear and Borel*.*

Example 2. Automatic continuity.
X Banach, Y normed space, $T : X \to Y$ linear and Borel. *Then T is continuous.*

Example 2. Automatic continuity.
X Banach, Y normed space, $T : X \to Y$ linear and Borel*. Then T is continuous.*

Exactly the same proof as the UBP.

Example 2. Automatic continuity.
X Banach, Y normed space, $T : X \to Y$ linear and Borel. *Then T is continuous.*

Exactly the same proof as the UBP.

Assume that T is not continuous.
$(x_n) \subseteq X$ such that $\|x_n\| \leq 2^{-n}$ and $\|T(x)\| \geq n$.
$\Phi_n(x) := \frac{1}{n}T(x) \to 0$ for all $x \in X$.
$\Phi_n(x_n) \not\to 0$.
Contradiction.

Example 3.

Example 3. Schur's ℓ_1 Theorem.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n\rangle| \geq \varepsilon > 0$

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports. (*Lazy gliding hump.*)

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n\rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports. (*Lazy gliding hump.*)

$\sum y_n^*$ is subseries convergent in $G := (\ell_\infty, w^*)$.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports. (*Lazy gliding hump.*)

$\sum y_n^*$ is subseries convergent in $G := (\ell_\infty, w^*)$.

$\Phi_n(y^*) := \langle y^*, y_n \rangle$

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n\rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports. (*Lazy gliding hump.*)

$\sum y_n^*$ is subseries convergent in $G := (\ell_\infty, w^*)$.

$\Phi_n(y^*) := \langle y^*, y_n\rangle \to 0$ for all $y^* \in G$.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n\rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports. (*Lazy gliding hump.*)

$\sum y_n^*$ is subseries convergent in $G := (\ell_\infty, w^*)$.

$\Phi_n(y^*) := \langle y^*, y_n\rangle \to 0$ for all $y^* \in G$.

$\Phi_n(y_n^*) \not\to 0$.

Example 3. Schur's ℓ_1 Theorem.

(x_n) sequence in ℓ_1 such that $x_n \xrightarrow{w} 0$.

Assume that $\|x_n\| \not\to 0$.

Then, one can find a subsequence (y_n) of (x_n) and a sequence $(y_n^*) \subseteq (\ell_1)^* = \ell_\infty$ with $\|y_n^*\| = 1$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and the y_n^* have finite, pairwise disjoint supports. (*Lazy gliding hump.*)

$\sum y_n^*$ is subseries convergent in $G := (\ell_\infty, w^*)$.

$\Phi_n(y^*) := \langle y^*, y_n \rangle \to 0$ for all $y^* \in G$.

$\Phi_n(y_n^*) \not\to 0$.

Contradiction.

Example 4.

Example 4. Nikodym's Boundedness Theorem.

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,*

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$*.*

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$*. Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$*.*

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| < \infty$.

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i\in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i\in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| = \infty$.

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ measurable space, $(\mu_i)_{i\in I}$ family of complex measure defined on $\mathfrak{T}$. Assume that $\sup_{i\in I} |\mu_i(E)| < \infty$ for every $E \in \mathfrak{T}$. Then $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$*. Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$*. Then* $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| < \infty$*.*

Assume that $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G :=$

Example 4. Nikodym's Boundedness Theorem.

$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$*. Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$*. Then* $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| < \infty$*.*

Assume that $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$,

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$*. Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$*. Then* $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| < \infty$*.*

Assume that $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$, topology of pointwise convergence on $M(\mathfrak{T})$.

Example 4. Nikodym's Boundedness Theorem.

$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i\in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i\in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}$-measurable simple functions on $X\}$, topology of pointwise convergence on $M(\mathfrak{T})$.

$\sum \mathbf{1}_{E_n}$ is subseries convergent in G.

Example 4. Nikodym's Boundedness Theorem.

$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i \in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i \in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i \in I} \sup_{E \in \mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$, topology of pointwise convergence on $M(\mathfrak{T})$.

$\sum \mathbf{1}_{E_n}$ is subseries convergent in G.

$\Phi_n(f) := \frac{1}{n} \int_X f \, d\mu_{i_n}$

Example 4. Nikodym's Boundedness Theorem.

$(X, \mathfrak{T})$ measurable space, $(\mu_i)_{i\in I}$ family of complex measure defined on $\mathfrak{T}$. Assume that $\sup_{i\in I} |\mu_i(E)| < \infty$ for every $E \in \mathfrak{T}$. Then $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$, topology of pointwise convergence on $M(\mathfrak{T})$.

$\sum \mathbf{1}_{E_n}$ is subseries convergent in G.

$\Phi_n(f) := \frac{1}{n} \int_X f \, d\mu_{i_n} \to 0$

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i\in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i\in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$, topology of pointwise convergence on $M(\mathfrak{T})$.

$\sum \mathbf{1}_{E_n}$ is subseries convergent in G.

$\Phi_n(f) := \frac{1}{n} \int_X f \, d\mu_{i_n} \to 0$ for all $f \in G$.

$\Phi_n(\mathbf{1}_{E_n}) \not\to 0$.

Example 4. Nikodym's Boundedness Theorem.
$(X, \mathfrak{T})$ *measurable space,* $(\mu_i)_{i\in I}$ *family of complex measure defined on* $\mathfrak{T}$. *Assume that* $\sup_{i\in I} |\mu_i(E)| < \infty$ *for every* $E \in \mathfrak{T}$. *Then* $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| < \infty$.

Assume that $\sup_{i\in I} \sup_{E\in\mathfrak{T}} |\mu_i(E)| = \infty$.

Then one can find $(i_n) \subseteq I$ and $E_n \in \mathfrak{T}$ such that $|\mu_{i_n}(E_n)| \geq n$ for all $n \in \mathbb{N}$ and the E_n are pairwise disjoint.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$, topology of pointwise convergence on $M(\mathfrak{T})$.

$\sum \mathbf{1}_{E_n}$ is subseries convergent in G.

$\Phi_n(f) := \frac{1}{n} \int_X f \, d\mu_{i_n} \to 0$ for all $f \in G$.

$\Phi_n(\mathbf{1}_{E_n}) \not\to 0$.

Contradiction.

Example 5.

Example 5. Vitali-Hahn-Saks Theorem.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,*

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$*.*

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} \mu_n(E)$ *exists for every* $E \in \mathfrak{T}$.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} \mu_n(E)$ *exists for every* $E \in \mathfrak{T}$. *Then* (μ_n) *is uniformly* μ*-continuous.*

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} \mu_n(E)$ *exists for every* $E \in \mathfrak{T}$. *Then* (μ_n) *is uniformly* μ*-continuous.*

Assume that (μ_n) is not uniformly μ-continuous.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$,

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} \mu_n(E)$ *exists for every* $E \in \mathfrak{T}$. *Then* (μ_n) *is uniformly* μ*-continuous.*

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.
WLOG

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

WLOG the E_n are pairwise disjoint and $|\nu_{n-1}(E_n)| < \varepsilon/2$.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$. *Assume that* $\lim_{n\to\infty} \mu_n(E)$ *exists for every* $E \in \mathfrak{T}$. *Then* (μ_n) *is uniformly* μ*-continuous.*

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

WLOG the E_n are pairwise disjoint and $|\nu_{n-1}(E_n)| < \varepsilon/2$. (*Lazy gliding hump again.*)

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

WLOG the E_n are pairwise disjoint and $|\nu_{n-1}(E_n)| < \varepsilon/2$. (*Lazy gliding hump again.*) Then $|\nu_n(E_n) - \nu_{n-1}(E_n)| > \varepsilon/2$.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ *measure space,* (μ_n) *sequence of complex measures defined on* $\mathfrak{T}$ *such that* $\mu_n << \mu$ *for all* $n \in \mathbb{N}$*. Assume that* $\lim_{n\to\infty} \mu_n(E)$ *exists for every* $E \in \mathfrak{T}$*. Then* (μ_n) *is uniformly* μ*-continuous.*

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

WLOG the E_n are pairwise disjoint and $|\nu_{n-1}(E_n)| < \varepsilon/2$. (*Lazy gliding hump again.*) Then $|\nu_n(E_n) - \nu_{n-1}(E_n)| > \varepsilon/2$.

Contradiction

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

WLOG the E_n are pairwise disjoint and $|\nu_{n-1}(E_n)| < \varepsilon/2$. (*Lazy gliding hump again.*) Then $|\nu_n(E_n) - \nu_{n-1}(E_n)| > \varepsilon/2$.

Contradiction as in the proof of Nikodym's Boundedness Theorem.

Example 5. Vitali-Hahn-Saks Theorem.
$(X, \mathfrak{T}, \mu)$ measure space, (μ_n) sequence of complex measures defined on $\mathfrak{T}$ such that $\mu_n << \mu$ for all $n \in \mathbb{N}$. Assume that $\lim_{n\to\infty} \mu_n(E)$ exists for every $E \in \mathfrak{T}$. Then (μ_n) is uniformly μ-continuous.

Assume that (μ_n) is not uniformly μ-continuous.

$(E_n) \subseteq \mathfrak{T}$, $(\nu_n) \preceq (\mu_n)$ such that $\mu(E_n) \to 0$ and $|\nu_n(E_n)| \geq \varepsilon > 0$.

WLOG the E_n are pairwise disjoint and $|\nu_{n-1}(E_n)| < \varepsilon/2$. (*Lazy gliding hump again.*) Then $|\nu_n(E_n) - \nu_{n-1}(E_n)| > \varepsilon/2$.

Contradiction as in the proof of Nikodym's Boundedness Theorem.

$G := \{\mathfrak{T}\text{-measurable simple functions on } X\}$,
$\Phi_n(f) := \int_X f\, d\nu_n - \int_X f\, d\nu_{n-1}$.

Example 6.

Example 6. Orlicz-Pettis Theorem.

Example 6. Orlicz-Pettis Theorem.

X *normed space,*

Example 6. Orlicz-Pettis Theorem.
X normed space, $(x_n) \subseteq X$.

Example 6. Orlicz-Pettis Theorem.
X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology,

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

Example 6. Orlicz-Pettis Theorem.
X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough:

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w),

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

• WLOG X is separable.

• Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

• WLOG X is separable.

• Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$

such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then ($x_n \xrightarrow{w} 0$ + separability) $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

• WLOG X is separable.

• Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then ($x_n \xrightarrow{w} 0$ + separability) $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

$\sum y_n$ is subseries convergent in $G := (X, w)$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

• WLOG X is separable.

• Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then ($x_n \xrightarrow{w} 0$ + separability) $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

$\sum y_n$ is subseries convergent in $G := (X, w)$.

$\Phi_n(x) := \langle y_n^*, x \rangle$

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

• WLOG X is separable.

• Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then ($x_n \xrightarrow{w} 0$ + separability) $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n\rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

$\sum y_n$ is subseries convergent in $G := (X, w)$.

$\Phi_n(x) := \langle y_n^*, x\rangle \to 0$ for all $x \in G$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

- WLOG X is separable.
- Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then ($x_n \xrightarrow{w} 0$ + separability) $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n \rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

$\sum y_n$ is subseries convergent in $G := (X, w)$.

$\Phi_n(x) := \langle y_n^*, x \rangle \to 0$ for all $x \in G$.

$\Phi_n(y_n) \not\to 0$.

Example 6. Orlicz-Pettis Theorem.

X normed space, $(x_n) \subseteq X$. If $\sum x_n$ is subseries convergent for the weak topology, then it is subseries convergent for the norm topology.

• WLOG X is separable.

• Enough: show that if $\sum x_n$ is subseries convergent in (X, w), then $\|x_n\| \to 0$.

Assume $\|x_n\| \not\to 0$.

Then ($x_n \xrightarrow{w} 0$ + separability) $\exists (y_n)$ subsequence of (x_n) and $(y_n^*) \subseteq X^*$ such that $|\langle y_n^*, y_n\rangle| \geq \varepsilon > 0$ and $y_n^* \xrightarrow{w^*} 0$.

$\sum y_n$ is subseries convergent in $G := (X, w)$.

$\Phi_n(x) := \langle y_n^*, x\rangle \to 0$ for all $x \in G$.

$\Phi_n(y_n) \not\to 0$.

Contradiction.