Homogeneous actions on the random graph

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The random graph

Consider a countable graph having the following property :

Property $(*)$

For any disjoint finite subsets of vertices U and V, there exist a vertex w such that $w \sim u, \forall u \in U$ and $w \nsim v, \forall v \in V$.

Up to isomorphism, there is a unique graph with this property, that is called the Random graph. We denote it $\mathcal{R}$.

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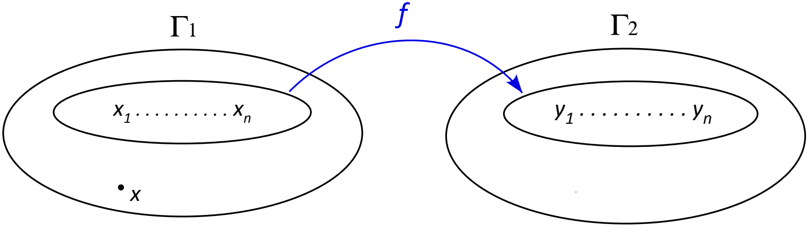

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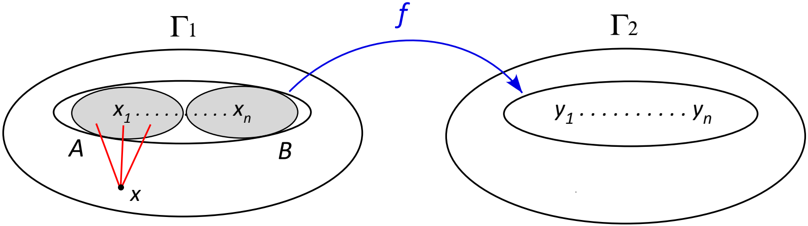

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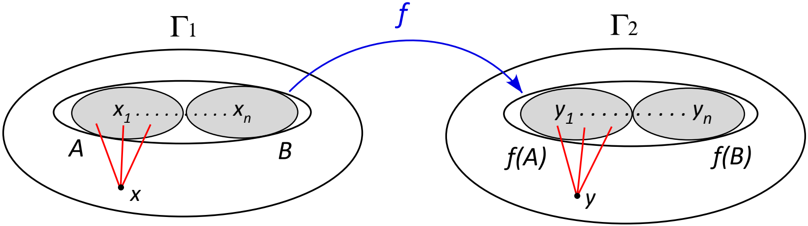

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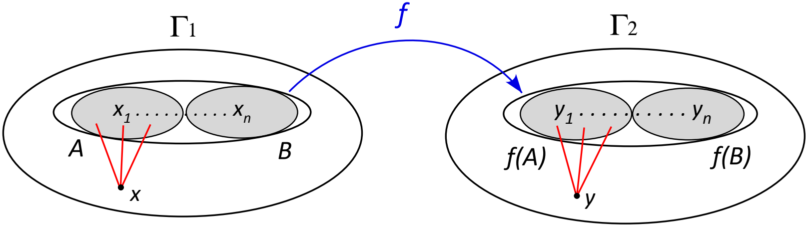

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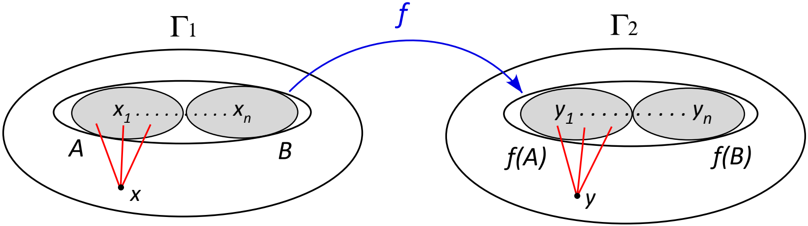

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Enumerate $\Gamma_1 = \{x_1, x_2, ...\}$, $\Gamma_2 = \{y_1, y_2, ...\}$; and build partial isomorphisms f_n by **"back-and-forth"** methode.

The random graph

Erdös-Rényi, 1963

If a countable graph is chosen at random, by selecting edges independently with probability $\frac{1}{2}$ from the set of 2-elements subsets of the vertex set, then almost surely the resulting graph is isomorphic to $\mathcal{R}$.

- (Rado's construction, 1964) $\mathcal{R}$ is universal, i.e. every at most countable graph can be embedded as an induced subgraph of $\mathcal{R}$.
- $\mathcal{R}$ is homogeneous, i.e. any graph isomorphism between finite induced subgraphs can be extended to a graph automorphism of $\mathcal{R}$.

The automorphism group of $\mathcal{R}$

$\mathrm{Aut}(\mathcal{R})$ is a closed subset of $\mathrm{Sym}(V(\mathcal{R}))$, thus is a Polish group (equipped with a topology of pointwise convergence).

Definition

We call a group action $\Gamma \curvearrowright \mathcal{R}$ homogeneous if, for any graph isomorphism $\varphi : U \to V$ between finite induced subgraphs U, V of $\mathcal{R}$, there exists $g \in \Gamma$ such that $g(u) = \varphi(u)$ for all $u \in U$.

- $\mathrm{Aut}(\mathcal{R}) \curvearrowright \mathcal{R}$ is homogeneous.
- A subgroup $G < \mathrm{Aut}(\mathcal{R})$ is dense if and only if the action $G \curvearrowright \mathcal{R}$ is homogeneous.

Dense subgroups of $\mathrm{Aut}(\mathcal{R})$

Let $\mathcal{H}_{\mathcal{R}}$ be the class of all countable groups that admit a faithful and homogeneous action on $\mathcal{R}$.

- Free groups (Macpherson 1986, Melles and Shelah 1994, Gartside and Knight 2003, Glaband and Strobin 2015)
- A locally finite group (Bhattacharjee and Macpherson, 2005)
- Any $\Gamma \in \mathcal{H}_{\mathcal{R}}$ must be icc and not solvable.

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Definition

A subgroup $\Sigma < \Gamma$ is called highly core-free if, for every finite subset $F \subset \Gamma$, for any $n \geq 1$, for any non-empty subsets $S_1, \ldots, S_n \subset \Gamma$ such that $\Gamma = \Sigma F \cup \cup_{k=1}^{n} S_k$ there exists $1 \leq k \leq n$ such that $\mathrm{Core}_{S_k}(\Sigma) = \{1\}$.

Example : finite subgroup of an icc group

Dense subgroups of $\mathrm{Aut}(\mathcal{R})$

Theorem A (Fima-M-Stalder, 2016)

If $\Sigma < \Gamma_1, \Gamma_2$ is a common finite subgroup such that Σ is highly core-free in Γ_1 and, either Γ_2 is infinite and Σ is highly core-free in Γ_2, or Γ_2 is finite and $[\Gamma_2 : \Sigma] \geq 2$, then $\Gamma_1 \underset{\Sigma}{*} \Gamma_2 \in \mathcal{H}_{\mathcal{R}}$.

Theorem B (Fima-M-Stalder, 2016)

Let H be an infinite countable group, $\Sigma < H$ a finite subgroup and $\theta : \Sigma \to H$ an injective group homomorphism. If both Σ and $\theta(\Sigma)$ are highly core-free in H then $\mathrm{HNN}(H, \Sigma, \theta) \in \mathcal{H}_{\mathcal{R}}$.

Corollary

Let Γ be a countable group acting, without inversion, on a non-trivial tree $\mathcal{T}$. If every vertex stabilizer of $\mathcal{T}$ is infinite and, for every edge e of $\mathcal{T}$ the stabilizer of e is finite and is a highly core-free subgroup of both the stabilizer of the source of e and the stabilizer of the range of e, then $\Gamma \in \mathcal{H}_{\mathcal{R}}$.

Idea of the proof

Let Γ_1, Γ_2 be two countable groups with a common finite subgroup Σ and define $\Gamma = \Gamma_1 \underset{\Sigma}{*} \Gamma_2 < \mathrm{Aut}(\mathcal{R})$. Let $Z := \{\alpha \in \mathrm{Aut}(\mathcal{R}) \,:\, \alpha\sigma = \sigma\alpha \ \forall \sigma \in \Sigma\}$. For all $\alpha \in Z$, $\exists!\ \pi_\alpha \,:\, \Gamma \to \mathrm{Aut}(\mathcal{R})$ such that:

$$\pi_\alpha(g) = \begin{cases} g & \text{if} \quad g \in \Gamma_1, \\ \alpha^{-1} g \alpha & \text{if} \quad g \in \Gamma_2. \end{cases}$$

Lemma

1. If $\Sigma \curvearrowright \mathcal{R}$ is free and non-singular and Σ is highly core-free w.r.t. $\Gamma_1, \Gamma_2 \curvearrowright \mathcal{R}$ then the set $U = \{\alpha \in Z \,:\, \pi_\alpha \text{ is homogeneous}\}$ is a dense G_δ in Z.
2. If Σ is highly core-free w.r.t. $\Gamma_1 \curvearrowright \mathcal{R}$, Γ_2 is finite and $\Gamma_2 \curvearrowright \mathcal{R}$ is free and non-singular such that $[\Gamma_2 : \Sigma] \geq 2$, then the set $U = \{\alpha \in Z \,:\, \pi_\alpha \text{ is homogeneous}\}$ is a dense G_δ in Z.
3. If $\Sigma \curvearrowright \mathcal{R}$ is free, $\Gamma \curvearrowright \mathcal{R}$ is non-singular and has property (F) then the set $V = \{\alpha \in Z \,:\, \pi_\alpha \text{ is faithful}\}$ is a dense G_δ in Z.

Idea of the proof

Construction of actions with properties that we want :

- $\Gamma \curvearrowright \mathcal{G}_0$ with $V(\mathcal{G}_0) = \Gamma$ and $E(\mathcal{G}_0) = \emptyset$ by left multiplication
- For $n \geq 0$, define
  $V(\mathcal{G}_{n+1}) = V(\mathcal{G}_n) \sqcup \mathcal{P}_f(V(\mathcal{G}_n))$
  $E(\mathcal{G}_{n+1}) = E(\mathcal{G}_n) \sqcup (\sqcup_{U \in \mathcal{P}_f(V(\mathcal{G}_n))} U \times \{U\}) \sqcup (\sqcup_{U \in \mathcal{P}_f(V(\mathcal{G}_n))} \{U\} \times U)$
- $\mathcal{G}_\infty := \cup \mathcal{G}_n$

Facts :

- $\mathcal{G}_\infty \simeq \mathcal{R}$
- Many properties of $\Gamma \curvearrowright \mathcal{G}_0$ pass to $\Gamma \curvearrowright \mathcal{G}_\infty$