Smooth convex extensions of jets from convex subsets of $\mathbb{R}^n$

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joint work with **Daniel Azagra**

Banach spaces and optimization:

Conference on the occasion of Robert Deville's 60th birthday

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If $C \subseteq \mathbb{R}^n$ is closed and $m \in \mathbb{N}$, a family of functions $\{A_0, A_1, \ldots, A_m\}$ defined on C such that $A_k(y) \in \mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$ for every $y \in C$ and every $k = 0, \ldots, m$ is a $\underline{m\text{-jet on } C}$.

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Theorem (Whitney extension theorem for C^m, 1934)

Let C be a closed subset of $\mathbb{R}^n$, and $m \in \mathbb{N}$. Let $\{A_k\}_{k=0}^m$ be a m-jet on C. Then, there exists a function $F \in C^m(\mathbb{R}^n)$ such that $D^k F = A_k$ on C for every $k = 0, \dots, m$ if and only if for every compact subset K of C and every $k = 0, \dots, m$

$$\lim_{|y-z| \to 0^+} \frac{\left\| A_k(z) - \sum_{\ell=0}^{m-k} \frac{1}{\ell!} A_{k+\ell}(y)(z-y)^\ell \right\|}{|y-z|^{m-k}} = 0 \text{ uniformly on } y, z \in K, \tag{W^m}$$

where $\|L\| = \sup_{v_1, \dots, v_k \in \mathbb{S}^{n-1}} |L(v_1, \dots, v_k)|$ for every $L \in \mathcal{L}^k(\mathbb{R}^n, \mathbb{R})$.

If $C \subseteq \mathbb{R}^n$ is closed, a family of functions $\{A_m\}_{m=0}^{\infty}$ defined on C is a ∞-jet on C if for every $m \in \mathbb{N}$ and every point $y \in C$, we have that $A_m(y) \in \mathcal{L}^m(\mathbb{R}^n, \mathbb{R})$.

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Theorem (Whitney extension theorem for C^∞, 1934)

Let C be a closed subset of $\mathbb{R}^n$. Let $\{A_m\}_{m \geq 0}$ be a ∞-jet on C. Then, there exists a function $F \in C^\infty(\mathbb{R}^n)$ such that $D^m F = A_m$ on C for every $m \in \mathbb{N}$ if and only if the subfamily $\{A_0, \dots, A_m\}$ satisfies (W^m) for every $m \in \mathbb{N}$.

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We will say that $\{A_m\}_{m\geq 0}$ satisfies (W^∞) provided that $\{A_0, \ldots, A_m\}$ satisfies (W^m) for every $m \in \mathbb{N}$.

Problem

Given $m \in \mathbb{N} \cup \{\infty\}$, an closed subset C of $\mathbb{R}^n$ and a m-jet $\{A_k\}_{k=0}^m$ defined on C, what necessary and sufficient conditions on $\{A_k\}_{k=0}^m$ would guarantee the existence of a **convex** function F of class $C^m(\mathbb{R}^n)$ such that $D^k F = A_k$ on C for every $0 \leq k \leq m$?

As a consequence of the results by M. Ghomi (2002) and M. Yan (2013):

If $C \subset \mathbb{R}^n$ is a compact convex body and $\{A_k\}_{k=0}^m$ is a m-jet on C satisfying (W^m) and such that $\underline{A_2(x) \text{ is positive definite for every } x \in \partial C}$, then we can find a convex function $F \in C^m(\mathbb{R}^n)$ with $D^k F = A_k$ on C for every k.

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Observe that positive definiteness of A_2 on ∂C is a very strong condition, far from being necessary.

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Theorem (D. Azagra and C. Mudarra, 2015)

Let C be a compact subset of $\mathbb{R}^n$. Let $(f,G): C \to \mathbb{R} \times \mathbb{R}^n$ be a 1-jet on C with G continuous. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F = f$ and $\nabla F = G$ on C if and only if (f,G) satisfies

$$(C) \qquad f(x) \geq f(y) + \langle G(y), x-y \rangle \text{ for all } x,y \in C;$$

$$(CW^1) \quad f(x) = f(y) + \langle G(y), x-y \rangle \implies G(x) = G(y) \text{ for all } x,y \in C.$$

Moreover, we can arrange $\mathrm{Lip}(F) \leq \kappa \sup_{y \in C} |G(y)|$, where κ is an absolute constant.

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We solved the problem when C is arbitrary (unbounded) as well, but the solution is much more complicated, due to the presence of *corners at infinity*.

Idea of the proof for C^1: Using those conditions, we construct a $C^1(\mathbb{R}^n)$ function g with $(g, \nabla g) = (f, G)$ on C, such that $\lim_{|x|\to\infty} g(x) = \infty$ and such that $g \geq m$, where

$$m(x) := \sup_{y\in C}\{f(y) + \langle G(y), x - y\rangle\}, \quad x \in \mathbb{R}^n.$$

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Then $F := \text{conv}(g)$ is the desired extension.

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• $g \in C^1(\mathbb{R}^n)$ + coercive $\implies$ $\text{conv}(g) \in C^1(\mathbb{R}^n)$ [Kirchheim-Kristensen, 2001]

• If $g \in C^\infty(\mathbb{R}^n)$, the best we can obtain in general is $\text{conv}(g) \in C^{1,1}(\mathbb{R}^n)$. For instance, if $g(t) = (t^2 - 1)^2,\ t \in \mathbb{R}$, then $F = \text{conv}(g)$ is the function given by

$$F(t) = \begin{cases} (t^2 - 1)^2 & \text{if } |t| \geq 1 \\ 0 & \text{if } t \in [-1, 1]; \end{cases}$$

and $F \in C^1(\mathbb{R}) \setminus C^2(\mathbb{R})$.

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But the function

$$F(x) = \begin{cases} \frac{1}{8}(x^2 - 1)^2 & \text{if } |x| \geq 1 \\ 0 & \text{if } x \in [-1, 1], \end{cases}$$

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$$\liminf_{t \to 0^+} \frac{1}{t^{m-2}} \left(A_2(y)(v^2) + \cdots + \frac{t^{m-2}}{(m-2)!} A_m(y)(w^{m-2}, v^2) \right) \geq 0$$

uniformly on $y \in C, w, v \in \mathbb{S}^{n-1}$.

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- (CW^m) is a necessary condition for C^m convex extensions (Taylor's theorem)
- (CW^2) says that $A_2(x)$ is positive semidefinite at every $x \in C$.
- $A_2 \succeq 0$ on C and $\{A_k\}_{k=0}^m$ (W^m) $\not\Longrightarrow$ $\{A_k\}_{k=0}^m$ (CW^m) even when C is a closed ball.
- Unfortunately, we have found examples showing that (CW^m) is not sufficient for C^m convex extension at least when C has empty interior.

C^m with $m \geq 2$?

Let $m \geq 2$ be an even integer and define

$$h(x,y) = \frac{(1-\cos(2\pi y))x^m}{2\pi}, \ (x,y) \in \mathbb{R}^2.$$

Let $C := \{0\} \times [0,1]$.

• $(h, Dh, \ldots, D^{m+1}h)$ satisfies conditions (W^{m+1}) and (CW^{m+1}) (and in particular (CW^m) too) on C.

• There is no convex function $F \in C^m(\mathbb{R}^2)$ such that $D^k F = D^k h$ on C for $k \in \{0, \ldots, m\}$.

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Idea of the proof: Since $\{A_m\}_{m\geq 0}$ satisfies (W^∞), we may find $f \in C^\infty(\mathbb{R}^n)$ (not necessarily convex) such that $D^m f(x) = A_m(x)$ for every $x \in C,\ m \in \mathbb{N}$. In particular, $D^2 f \geq 0$ on C.

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First step: We estimate the *lack of convexity* of f outside C: by using the condition (CW^∞) and a partition of unity of $[0, +\infty)$ we can find a function $\varepsilon : \mathbb{R} \to [0, +\infty)$ of class C^∞ such that $\varepsilon^{-1}(0) = (-\infty, 0]$ and

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Second step: We construct a nonnegative convex function $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi^{-1}(0) = C,\ D^m \varphi = 0$ on C for every m, and

$$D^2 \varphi(x)(v^2) \geq 2\varepsilon\left(d(x, C)\right) \quad \text{for all} \quad x \in \mathbb{R}^n \setminus C,\ v \in \mathbb{S}^{n-1}.$$

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Then $F := f + \varphi$ is the desired function.

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Construction of φ. We consider a nonnegative function $g \in C^\infty(\mathbb{R})$ such that $g^{-1}(-\infty, 0] = \{0\}$ and g'' satisfies "good lower estimates" depending on ε.

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We also consider the *support function* of C given by $h(w) = \max_{z \in C} \langle z, w \rangle$ for every $w \in \mathbb{S}^{n-1}$. We define the function φ by

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} g\left(\langle x, w \rangle - h(w)\right) dw, \quad x \in \mathbb{R}^n.$$

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Given $x \in \mathbb{R}^n \setminus C$ and $v \in \mathbb{S}^{n-1}$, we find a region $W = W(x, v) \subset \mathbb{S}^{n-1}$ such that $\mathrm{vol}_{\mathbb{S}^{n-1}}(W)$ depends only on $d(x, C)$ and we have good lower estimates on $\langle x, w\rangle - h(w)$ and $\langle v, w\rangle$ whenever $w \in W$.

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If $\varepsilon \in C^\infty \implies g \in C^\infty$. And then $\varphi \in C^\infty(\mathbb{R}^n)$.

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More precisely: $\langle x, w\rangle - h(w) \approx d(x, C)$ and $\langle v, w\rangle \gtrsim d(x, C)$ if $w \in W$.

We thus have $D^2\varphi(x)(v^2) \gtrsim \mathrm{vol}_{\mathbb{S}^{n-1}}(W) g''(d(x, C)) d(x, C)^2$

The set W is a spherical cap and then $\mathrm{vol}_{\mathbb{S}^{n-1}}(W) \approx d(x, C)^{n-1}$... therefore we need g to satisfy $g''(t) \gtrsim \varepsilon(t)/t^{n+1}$. For instance,

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If $\varepsilon \in C^\infty \implies g \in C^\infty$. And then $\varphi \in C^\infty(\mathbb{R}^n)$.

But if $m < \infty$, we will obtain that $D^2 f(x)(v^2) \geq -\varepsilon(d(x, C))$, where $\varepsilon \in C^{m-2}(\mathbb{R})$

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But if $m < \infty$, we will obtain that $D^2 f(x)(v^2) \geq -\varepsilon(d(x, C))$, where $\varepsilon \in C^{m-2}(\mathbb{R}) \implies g \in C^{m-n-1} \implies \varphi \in C^{m-n-1}(\mathbb{R}^n)$.

C^m with $m \geq 2$?

Theorem

Let C be a compact convex subset of $\mathbb{R}^n$. Let $m \in \mathbb{N}$ with $m \geq n+3$, and let $\{A_k\}_{k=0}^m$ be a m-jet on C. Assume that $\{A_k\}_{k=0}^m$ satisfies conditions (W^m) and (CW^m) on C. Then there exists a convex function $F \in C^{m-n-1}(\mathbb{R}^n)$ such that $D^kF(x) = A_k(x)$ for every $x \in C$, and every $k = 0, \ldots, m-n-1$.

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Theorem

Let C be the intersection of a finite family of ovaloids of class C^{m-1}, let $m \geq 3$ and let $\{A_k\}_{k=0}^m$ be a m-jet on C. Assume that $\{A_k\}_{k=0}^m$ satisfies satisfies (W^m) and (CW^m) on C. Then there exists a convex function $F \in C^{m-1}(\mathbb{R}^n)$ such that $D^k F(x) = A_k(x)$ for every $x \in C$ and every $k = 0, \ldots, m-1$.

Thank you very much!