# Symmetric strong diameter two property

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# Introduction

This talk is based on the joint paper [HLLN].
We only look at non-trivial real Banach spaces. Throughout, let $X$ be a Banach space.
A slice of $B_X$ is a set of the form

$$S(x^*, \alpha) = \{x \in B_X \mid x^*(x) > 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha > 0$.

# Strong diameter two property and almost squareness

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## Definition ([ALL])

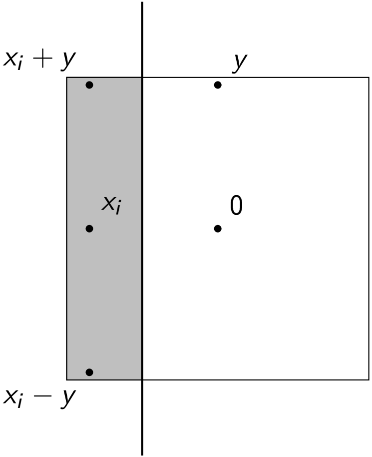
*A Banach space $X$ is almost square (ASQ), if whenever $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in S_X$, there exists a sequence $(y_k)_{k=1}^{\infty} \subset S_X$ such that $\|x_i \pm y_k\| \to_k 1$ for every $i \in \{1, \ldots, n\}$.*

# Symmetric strong diameter two property

## Definition ([ALN], [ANP])

*A Banach space $X$ has the symmetric strong diameter two property (SSD2P), if for every finite family $\{S_i\}_{i=1}^{n}$ of slices of $B_X$ and $\varepsilon > 0$, there exist $x_i \in S_i$ and $y \in B_X$, independent of $i$, such that $x_i \pm y \in S_i$ for every $i \in \{1, \ldots, n\}$ and $\|y\| > 1 - \varepsilon$.*

# A picture!




$$X = \ell^2_\infty$$

# SSD2P $\Rightarrow$ SD2P

Let $S_1, \ldots, S_n$ be slices of $B_X$ and $S = \sum_{i=1}^n \lambda_i S_i$ be their convex combination. Due to SSD2P we have $x_i \in S_i$ and $y \in B_X$ such that $\|y\| > 1 - \varepsilon$ and $x_i \pm y \in S_i$.
Then we have

$$diam(S) \geq \|\sum_{i=1}^n \lambda_i(x_i + y) - \sum_{i=1}^n \lambda_i(x_i - y)\| = 2\|y\| \geq 2 - 2\varepsilon.$$

# Absolute norms

A norm is called *absolute* if

$$N(a,b) = N(|a|,|b|) \quad \text{for all } (a,b) \in \mathbb{R}^2$$

and *normalised* if $N(1,0) = N(0,1) = 1$.
If $X$ and $Y$ are Banach spaces, then denote $X \oplus_N Y$ the product space $X \times Y$ with respect to the norm $\|(x,y)\|_N = N(\|x\|,\|y\|)$ for all $x \in X$ and $y \in Y$.

# SD2P $\not\Rightarrow$ SSD2P

**Theorem ([HLLN])**

*The $\infty$-norm is the only absolute normalised norm $N$ which allows SSD2P on $X \oplus_N Y$.*

$L_1[0,1]$ has the SD2P but $L_1[0,\frac{1}{2}] \oplus_1 L_1[\frac{1}{2},1]$ fails the SSD2P.

# ASQ $\Rightarrow$ SSD2P

## Propostion ([HLLN])

*A Banach space $X$ has the SSD2P if and only if whenever $n \in \mathbb{N}$, $x_1, \ldots, x_n \in S_X$, there exist nets $(y_{i.\alpha}) \subset S_X$ and $(z_\alpha) \subset S_X$ such that $y_{i,\alpha} \to_\alpha x_i$ weakly, $z_\alpha \to 0$ weakly, and $\|y_{i,\alpha} \pm z_\alpha\| \to 1$ for all $i \in 1, \ldots, n$.*

# SSD2P $\not\Rightarrow$ ASQ

$C[0,1]$ has the SSD2P. Let $f_1, \dots, f_n \in S_{C[0,1]}$, $x_1, \dots, x_n \in [0,1]$ such that $|f_i(x_i)| = 1$ and $a > 0$ such that $2an < 1$. Let $(U_k)_{k\in\mathbb{N}}$ be pairwise disjoint and from the set

$$[0,1] \setminus \cup_{i=1}^{n} (x_i - a, x_i + a).$$

Choose $g_k \geq 0$ such that $\|g_k\| = 1$ and $\operatorname{supp} g_k \subset U_k$. Then $g_k \to^{w} 0$. Define $y_{i,k} = (1 - g_k) f_i$ and $z_k = g_k$.

# SSD2P $\not\Rightarrow$ ASQ

$C[0,1]$ is not ASQ. Consider the constant function $f = 1$. Then for every $(g_k) \subset S_{C[0,1]}$ we have

$$\|f \pm g_k\| = \max_{x\in[0,1]} |1 \pm g_k(x)|.$$

So

$$\max\{\|f + g_k\|, \|f - g_k\|\} = 2.$$

# w*-SSD2P and $\mathrm{Lip}_0(M)$

## Definition ([HLLN])

*A dual Banach space $X^*$ has the weak* symmetric strong diameter two property (w*-SSD2P) if for every finite family of $\{S_i\}_{i=1}^n$ of weak* slices of $B_{X^*}$ and $\varepsilon > 0$ there exists $x_i^* \in S_i$ and $y^* \in B_{X^*}$ such that $x_i^* \pm y^* \in S_i$ for every $i \in \{1, \ldots, n\}$ and $\|y^*\| > 1 - \varepsilon$.*

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Let $M$ be a pointed metric space with the metric $d$ and the origin designated as 0. The space $\mathrm{Lip}_0(M)$ of all Lipschitz functions $f : M \to \mathbb{R}$ with $f(0) = 0$ is a Banach space with the norm

$$\|f\| = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in M, x \neq y\right\}.$$

# Some recent progress

For each $n \in \mathbb{N}$ define

$$K_n = \{x \in \ell_\infty \mid x(k) \in \{0, 1, \dots, n\}\}.$$

From [HLLN]: $Lip_0(K_n)$ has the w*-SSD2P for every $n \in \mathbb{N}$.

From [LRZ]: $Lip_0(K_n)$ has the SSD2P for every $n \in \mathbb{N} \setminus \{2\}$.

## Thank you for your attention!


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