On Ordinal Ranks of Baire Class Functions

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(joint work with Denny Ho-Hon Leung [2] and Wee-Kee Tang [1])

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Outline

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Baire Class Functions

X Polish space (separable complete metric space).

Definition.

$f : X \to \mathbb{R}$ is said to be **Baire Class** 1 *if it is a pointwise limit of a sequence of continuous functions.*

$\xi \geq 1$ countable ordinal

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$f : X \to \mathbb{R}$ is said to be **Baire Class** ξ *if it is a pointwise limit of a sequence of Baire Class ζ functions where $\zeta < \xi$*

Remark: f is Baire Class 0 if it is continuous

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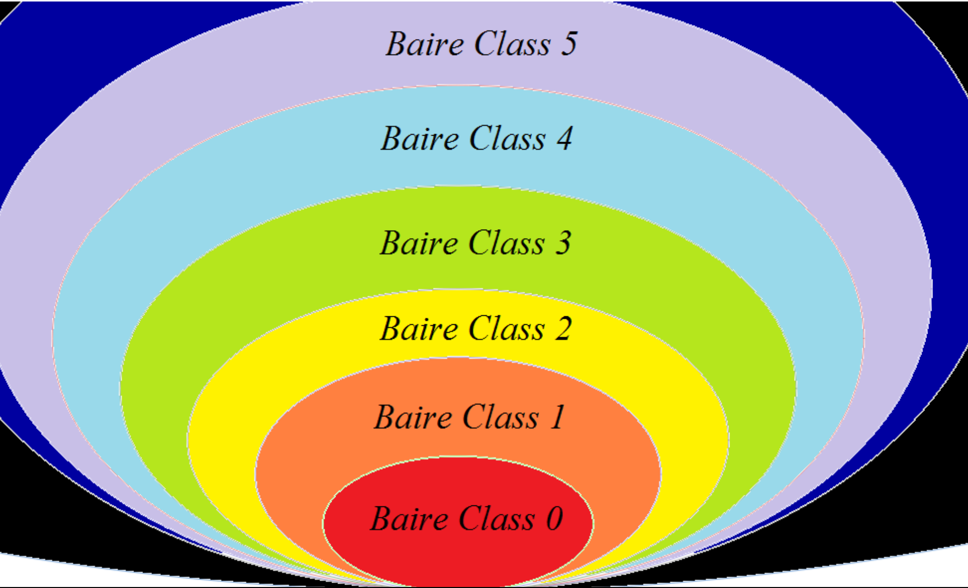
Baire Hierarchy



Equivalent Definitions of Baire Class 1 functions

X Polish space (separable complete metric space)

Theorem.

$f : X \to \mathbb{R}$ *is* **Baire Class** 1 *if it satisfies one of the following.*

1. $f_{|F}$ *has a point of continuity for every non-empty closed subset* $F \subseteq X$
2. f *is a pointwise limit of continuous functions*
3. $f^{-1}(U)$ *is* F_σ *for every open* $U \subseteq X$

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$f : X \to \mathbb{R}$ *is* **Baire Class** 1 *if it satisfies one of the following.*

1. $f_{|F}$ *has a point of continuity for every non-empty closed subset* $F \subseteq X$ *(oscillation rank)*
2. f *is a pointwise limit of continuous functions (convergence rank)*
3. $f^{-1}(U)$ *is* F_σ *for every open* $U \subseteq X$ *(separation rank)*

A. S. Kechris and A. Louveau, A classification of Baire 1 functions, Trans. Amer. Math. Soc. **318** (1990), 209-236.

Oscillation Rank β

$f : X \to \mathbb{R}$ is a Baire Class 1 function and $\varepsilon > 0$. Recall that

$$osc_\alpha(f,x) = \inf_{\substack{x\in U \\ U \text{ open}}} \sup_{y,z\in U\cap D^\alpha} |f(y)-f(z)|.$$

$D^0 = X$

$D^1 = \{x \in X : osc_0(f,x) \geq \varepsilon\}$ (ε-discontinuities)

$D^2 = \{x \in D^1 : osc_1(f,x) \geq \varepsilon\}$ (limit points of ε- discontinuities)

$D^3 = \{x \in D^2 : osc_2(f,x) \geq \varepsilon\}$ (limit points of D^2)

$\vdots$

$D^\omega = \cap_{n=1}^\infty D^n$

$\beta(f,\varepsilon) = \min\{\eta : D^\eta = \emptyset\} \geq 1$ $\beta(f) = \sup\{\beta(f,\varepsilon) : \varepsilon > 0\}$

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Examples of β

Recall that $f : X \to \mathbb{R}$,

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Examples

1. f is continuous $\Rightarrow \beta(f) = 1$
2. $X = \mathbb{R}$

$$A = \{0\} \Rightarrow \beta(\chi_A) = 2$$

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \left\{\frac{1}{n}\right\} \Rightarrow \beta(\chi_A) = 3$$

$$A = \{0\} \cup \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \left(\left\{\frac{1}{n}\right\} \cup \left\{\frac{1}{n} + \frac{1}{m}\right\}\right) \Rightarrow \beta(\chi_A) = 4$$

3. f is Baire Class 1 $\Rightarrow \beta(f) < \omega_1$

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Convergence Rank γ

Assume that $(f_n)_{n\in\mathbb{N}}$ sequence of functions on X and $\varepsilon > 0$

$$NUC_\alpha((f_n),x) = \inf_{\substack{x\in U\\ U \text{ open}}} \inf_{N\in\mathbb{N}} \sup\{|f_m(y)-f_n(y)| : y \in U\cap D^\alpha, n,m \geq N\}$$

$$D^0 = X$$

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Generalized β_2^* and γ_2^* Ranks

Assume: $\mathcal{F}_\sigma(\tau)$ is a collection of all F_σ sets with respect to τ, (X,τ) is a Polish space, $f:(X,\tau)\to\mathbb{R}$ Baire Class 2 function

Define

$$T_{f,2}=\{\tau':\tau' \text{ is Polish },\tau\subseteq\tau'\subseteq\mathcal{F}_\sigma(\tau), f\in\mathfrak{B}_1(\tau')\}$$

Definition (Elekes, Kiss, Vidnyánszky 16').

$$\beta_2^*(f)=\min\{\beta_{\tau'}(f):\tau'\in T_{f,2}\}$$
$$\gamma_2^*(f)=\min\{\gamma_{\tau'}(f):\tau'\in T_{f,2}\}$$

where $\beta_{\tau'}(f)$ and $\gamma_{\tau'}(f)$ are $\beta(f)$ and $\gamma(f)$ in τ' topology.

Márton Elekes, Viktor Kiss and Zoltán Vidnyánszky, Ranks on the Baire class ξ functions, Trans. Amer. Math. Soc. **368**(2016), 8111-8143.

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Kuratowski's Theorem

$$T_{f,2} = \{\tau' : \tau' \text{ is Polish }, \tau \subseteq \tau' \subseteq \mathcal{F}_\sigma(\tau), f \in \mathfrak{B}_1(\tau')\} \neq \emptyset?$$

Theorem (Kuratowski).

Assume that

1. (X, τ) *Polish space*
2. $A \in \mathcal{F}_\sigma(\tau) \cap \mathcal{G}_\delta(\tau)$

Then there is a Polish topology τ' *such that*

1. $\tau \subseteq \tau' \subseteq \mathcal{F}_\sigma(\tau)$
2. A *is clopen in* τ'

Recall that

1. $\mathcal{F}_\sigma(\tau)$ is a collection of all F_σ sets with respect to τ
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Solved Problems

For any $f, g \in \mathfrak{B}_2$,

1. $\beta_2^* \approx \gamma_2^*$ (essentially equivalent) if

$$\beta_2^*(f) \lesssim \gamma_2^*(f) \quad \text{and} \quad \gamma_2^*(f) \lesssim \beta_2^*(f)$$

2. β_2^* is essentially multiplicative if

$$\beta_2^*(fg) \leq \max\{\beta_2^*(f), \beta_2^*(g)\}$$

Problem (Elekes, Kiss, Vidnyánszky 16').

1. *$\beta_2^* \approx \gamma_2^*$ on $\mathfrak{B}_2$?*
2. *Are the ranks β_2^* and γ_2^* essentially multiplicative?*

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Main results

Theorem (Leung, Ng, Tang 19').

$\beta_2^* \approx \gamma_2^*$ *on* $\mathfrak{B}_2$

Theorem (Leung, Ng, Tang 19').

Both β_2^* *and* γ_2^* *are not essentially multiplicative.*

Outline

Open Problem

Proposition.

Assume that $\zeta \geq 1$ is an ordinal. Then

1. ζ *countable ordinal* $\Rightarrow$ *there exists* $\chi_A \in \mathfrak{B}_\xi(\tau)$ *such that* $\zeta < \beta^*_\xi(\chi_A) \leq \zeta + 2$.
2. ζ *limit ordinal* $\Rightarrow$ *there exists* $f \in \mathfrak{B}_\xi(\tau)$ *such that* $\beta^*_\xi(f) = \zeta$.

Problem.

Assume that

1. (X, τ) *be an uncountable Polish space*
2. $\xi \geq 2$ *be a countable ordinal.*

Is it true that for any nonzero countable ordinal ζ, *there exists* $f \in \mathfrak{B}_\xi(\tau)$ *such that* $\beta^*_\xi(f) = \zeta$?

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