Renormings through Deville's Master Lemma

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The coauthors

- S. Troyanski and J.O. *Deville's Master Lemma and Stone's discreteness in Renorming Theory* J. Convex Analysis 16 954–972 (2009).
- S. Troyanski and J.O. *LUR renormings through Deville's master Lemma* RACSAM 103 (1), (2009) 75–85.
- R. Smith, S. Troyanski and J. O. *Strictly convex norms and topology* Proc. London Math. Soc. 104(3) 197-222 (2012).
- J.O. *On $\mathcal{T}_p$ locally uniformly rotund norms* Set Valued and Variational Analysis, (2013).
- S. Ferrari, L. Oncina, M. Raja and J.O. *Metrization theory and the Kadec property.* Banach J. Math. Anal. 10 (2016), no. 2, 281–306.
- S. Ferrari, M. Raja and J.O.*Metrizability of Spheres and Renormings of Banach spaces* Quarterly J. Math 67(1) (2016) 15–27.
- S. Ferrari, M. Raja and J.O. *Generalized metric properties of spheres and renormings of Banach spaces* RACSAM

Deville's Master Lemma

$(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ are families of non-negative convex functions which are uniformly bounded on bounded subsets of a Banach space X. For every $x \in X$, $m \in \mathbb{N}$ and $i \in I$ let us define
$\varphi(x) := \sup\{\varphi_i(x) : i \in I\}$, $\theta_{i,m}(x) := \varphi_i(x)^2 + \frac{1}{m}\psi_i(x)^2$,
$\theta_m(x) := \sup\{\theta_{i,m}(x) : i \in I\}$, and

$$\theta(x) := \|x\|^2 + \sum_{m=1}^{\infty} 2^{-m}(\theta_m(x) + \theta_m(-x)).$$

Then the Minkowski functional of $B = \{x \in X : \theta(x) \leq 1\}$ is an equivalent norm $\|\cdot\|_B$ on X such that if $x_n, x \in E$ satisfy the LUR condition:

$$\lim_n [2\|x_n\|_B^2 + 2\|x\|_B^2 - \|x_n + x\|_B^2] = 0,$$

then there is a sequence (i_n) in I with the properties:

1. $\lim_n \varphi_{i_n}(x) = \lim_n \varphi_{i_n}(x_n) = \lim_n \varphi_{i_n}((x + x_n)/2) = \sup\{\varphi_i(x) : i \in I\}$
2. $\lim_n \left[\frac{1}{2}\psi_{i_n}^2(x_n) + \frac{1}{2}\psi_{i_n}^2(x) - \psi_{i_n}^2\left(\frac{1}{2}(x_n + x)\right)\right] = 0.$

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Slice Convex localization

Theorem

Let A be a bounded subset of X and $\mathcal{H}$ a family of $\sigma(X, F)$-open half space. Given a uniformly bounded on bounded sets family

$$\{\psi_H^A : E \to [0, +\infty) : H \in \mathcal{H}\}$$

of convex and $\sigma(X, F)$-lower semicontinuous functions, then there is an equivalent $\sigma(X, F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ such that: If $x \in A \cap H$ for some $H \in \mathcal{H}$ and (x_n) is a sequence in X such that $\lim_n \left(2\|x_n\|^2_{\mathcal{H},A} + 2\|x\|^2_{\mathcal{H},A} - \|x + x_n\|^2_{\mathcal{H},A}\right) = 0$, then there is a sequence of open half spaces $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ satisfying:

1. *There is $n_0 \in \mathbb{N}$ such that $x, x_n \in H_n$ for $n \geq n_0$ if $x_n \in A$.*
2. $\lim_{n\to\infty} \left[\frac{1}{2}\psi^A_{H_n}(x_n)^2 + \frac{1}{2}\psi^A_{H_n}(x)^2 - \psi^A_{H_n}(\frac{x+x_n}{2})^2\right] = 0.$

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Theorem (S. Troyanski and J.O)

Let A be a bounded subset in X and $\mathcal{H}$ a family of $\sigma(X, F)$-open half spaces.
Then there is an equivalent $\sigma(X, F)$-lower semicontinuous norm $\|\cdot\|_{\mathcal{H},A}$ such that: for every sequence (x_n) in X and $x \in A \cap H$ for some $H \in \mathcal{H}$, the condition $\lim_n \left(2\|x_n\|^2_{\mathcal{H},A} + 2\|x\|^2_{\mathcal{H},A} - \|x + x_n\|^2_{\mathcal{H},A}\right) = 0$ implies that there is a sequence of open half spaces $\{H_n \in \mathcal{H}\}_n$ such that:

1. *There is $n_0 \in \mathbb{N}$ such that $x \in H_n$ and $x_n \in H_n$ if $x_n \in A$ for $n \geq n_0$.*
2. *For every $\delta > 0$ there is some n_δ such that $x, x_n \in \overline{(co(A \cap H_n) + B(0, \delta))}^{\sigma(E,F)}$ for all $n \geq n_\delta$.*

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Quasi-denting LUR renorming theorem

Theorem (F. Garcia, S. Troyanski and J.O.)

Let X be a normed space with a norming subspace F in X^*. If there is a sequence of sets (A_n) such that for every $x \in X$ and every $\epsilon > 0$ there is $p \in N$ and a $\sigma(X, F)$-open half space H such that $x \in H \cap A_p$ is not empty and can be covered by finitely many sets of diameter less than ϵ, then X admits an equivalent $\sigma(E, F)$-lower semicontinuous and ***LUR*** *norm.*

Lemma

Let A be a subset of X, $\epsilon > 0$ and $\mathcal{H}$ a family of $\sigma(E,F)$-open half spaces such that for every $H \in \mathcal{H}$ the slice $H \cap A$ is not empty and covered by finitely many sets of diameter less than ϵ. Then there is a family $\{\psi_H : H \in \mathcal{H}\}$ of non-negative, convex and $\sigma(E,F)$-lower semicontinuous functions such that, given sequences $(x_n) \subset X$ and $\{H_n \in \mathcal{H} : n = 1, 2, ...\}$ with $x \in A \cap H_n$, for every $n \in \mathbb{N}$, it follows that

$$\|x_n - x\| < 3\epsilon$$

for n big enough, whenever we have

$$\lim_n \left[(\frac{1}{2}\psi_{H_n}(x_n)^2 + \frac{1}{2}\psi_{H_n}(x)^2 - \psi_{H_n}(\frac{x_n + x}{2})^2 \right] = 0.$$

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Theorem (J.F. Martínez, A. Moltó, S. Troyanski and J.O.)

Let $K \subset [0,1]^\Gamma$ be a compact space such that there is a sequence of sets (A_n) in $C(K)$ with the property that, for every $x \in C(K)$ and every $\epsilon > 0$ there exist $p \in N$ and a pointwise open half space H, together with a finite subset $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ of coordinates in Γ, such that $x \in H \cap A_p$, and for every $y \in A \cap H_p$ there exists $\delta_y > 0$ so that

$$|y(s) - y(t)| < \epsilon,$$

whenever $|s(\gamma_i) - t(\gamma_i)| < \delta_y$ for $i = 1, 2, \ldots, N$.
Then $C(K)$ admits a pointwise lower semicontinuous equivalent **LUR** *norm.*

Corollary

$C(K)$ admits a pointwise lower semicontinuos **LUR** *norm in the following cases:*

1. *K is σ-discrete. (R. Haydon)*
2. *K is the w^* dual unit ball of a dual Banach space with a dual* **LUR** *norm. (R. Haydon)*
3. *$K \subset [0,1]^P$ is separable, where P is a Polish space and every $s \in K$ has at most countably many discontinuities. (R. Haydon, A. Moltó and J.O.)*

LUR-Basis

Theorem (S. Troyanski and J.O)

*Let X be a normed space with a norming subspace $F \subset X^*$. X admits an equivalent $\sigma(X, F)$-lower semicontinuous and **LUR** norm if, and only if, the norm topology admits a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that every one of the families $\mathcal{B}_n$ is $\sigma(X, F)$-slicely isolated and norm discrete.*

Theorem

*Let X^* be dual Banach space. X^* admits an equivalent dual and **LUR** norm if, and only if, the norm topology admits a basis $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ such that each family $\mathcal{B}_n$ is relatively weak*-discrete in its union and norm discrete on the whole space E^*.*

Definition

We say that a topological space (X, τ) is a $T_0(*)$-space, (resp. is an $T_1(*)$-space) if there are families of open sets $\mathcal{W}_n$, $n = 1, 2, \ldots$, such that for $x \neq y$ there are some $p \in \mathbb{N}$ and either we have $y \notin \mathrm{Star}(x, \mathcal{W}_p) \neq \emptyset$ or $x \notin \mathrm{Star}(y, \mathcal{W}_p) \neq \emptyset$ (resp. $x \notin \mathrm{Star}(y, \mathcal{W}_p) \neq \emptyset$)).

Theorem (R. Smith, S. Troyanski and J.O.)

Let X be a normed space with a norming subspace $F \subset x^*$. Then X admits an equivalent $\sigma(X, F)$-lower semicontinuous and strictly convex norm if, and only if, there are families $\mathcal{H}_n$, $n = 1, 2, \ldots$, of $\sigma(x, F)$-open half spaces that $T_0(*)$ separates points of E.

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Theorem (R. Smith, S. Troyansky and J.O.)

For a scattered compact space K to be a $T_0(*)$-space is equivalent to have an equivalent dual strictly convex norm on the dual space $C(K)^*$

Theorem (S. Ferrari, M. Raja and J.O.)

Let X^* be a dual Banach space. It admits an equivalent dual strictly convex norm if, and only, if it has an equivalent dual unit sphere S_{X^*} such that the diagonal is a $\mathcal{G}_\delta$ subset of the product $(S_{X^*}, \sigma(X^*, X)) \times (S_{X^*}, \sigma(X^*, X))$

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Pointwise LUR renormings

Theorem

Let X be a normed space with a norming subspace $F \subset X^*$. The following statements are equivalent:

1. *X admits an equivalent $\sigma(X,F)$-lower semicontinuous and $\sigma(X,F)$-**LUR** norm.*
2. *$\sigma(X,F)$ admits a network $\mathcal{N} = \bigcup_{n=1}^{\infty} \mathcal{N}_n$ where $\mathcal{N}_n$ is $\sigma(X,F)$-slicely isolated for every $n \in \mathbb{N}$.*
3. *There are families $\mathcal{H}_n$ of $\sigma(X,F)$-open half spaces and non void sets $A_p \subset E$ such that*

$$\{\mathrm{Star}(x, \mathcal{H}_n) \cap A_p : n, p \in \mathbb{N}\}$$

is a network of the $\sigma(X,F)$-topology on $X \setminus \{0\}$.

Multiple-slice Localization Theorem

Theorem

Let X be a normed subspace of $\ell^\infty(\Gamma)$. Given sequences $(A_p)_{p=1}^\infty$ and $(\mathcal{H}_n)_{n=1}^\infty$ of bounded subsets A_p of X and families $\mathcal{H}_n$ of T_p-open half spaces respectively, there is an equivalent T_p-lower semicontinuous norm $\|\cdot\|_0$ such that, for every finite selection of pairs of positive integers $(m_1, p_1), (m_2, p_2), \cdots, (m_r, p_r)$, every $x \in \cap_{j=1}^r A_{p_j} \cap \bigcup \mathcal{H}_{m_j}$, for some $H_0^j \in \mathcal{H}_{m_j}, j = 1, 2, \cdots, r$ and any sequence (x_n) in X with $\lim_n (2\|x_n\|_0^2 + 2\|x\|_0^2 - \|x + x_n\|_0^2) = 0$, it follows that there are sequences of T_p-open half spaces $\{(H_n^1, \ldots, H_n^r) \in \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r}, n = 1, 2 \ldots\}$ such that

1. *There is $n_0 \in \mathbb{N}$ with $x \in \cap_{j=1}^r H_n^j$ and $x_n \in H_n^j$ if $x_n \in A_{p_j}$ for any $j = 1, \ldots, r$ whenever $n \geq n_0$ and*
2. *For every $\delta > 0$ there is some n_δ such that $x, x_n \in \overline{(\mathrm{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0, \delta))}^{T_p}$ for all $n \geq n_\delta$.*

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Let X be a normed subspace of $\ell^\infty(\Gamma)$. Given sequences $(A_p)_{p=1}^\infty$ and $(\mathcal{H}_n)_{n=1}^\infty$ of bounded subsets A_p of X and families $\mathcal{H}_n$ of T_p-open half spaces respectively, there is an equivalent T_p-lower semicontinuous norm $\|\cdot\|_0$ such that, for every finite selection of pairs of positive integers $(m_1, p_1), (m_2, p_2), \cdots, (m_r, p_r)$, every $x \in \cap_{j=1}^r A_{p_j} \cap \bigcup \mathcal{H}_{m_j}$, for some $H_0^j \in \mathcal{H}_{m_j}, j = 1, 2, \cdots, r$ and any sequence (x_n) in X with $\lim_n (2\|x_n\|_0^2 + 2\|x\|_0^2 - \|x + x_n\|_0^2) = 0$, it follows that there are sequences of T_p-open half spaces $\{(H_n^1, \ldots, H_n^r) \in \mathcal{H}_{m_1} \times \cdots \times \mathcal{H}_{m_r}, n = 1, 2 \ldots\}$ such that

1. *There is $n_0 \in \mathbb{N}$ with $x \in \cap_{j=1}^r H_n^j$ and $x_n \in H_n^j$ if $x_n \in A_{p_j}$ for any $j = 1, \ldots, r$ whenever $n \geq n_0$ and*
2. *For every $\delta > 0$ there is some n_δ such that $x, x_n \in \overline{(\mathrm{co}(\cap_{j=1}^r A_{p_j} \cap H_n^j) + B(0, \delta))}^{T_p}$ for all $n \geq n_\delta$.*

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My Birthday Theorem for Robert

Theorem

Let X be a normed space with a norming subspace F in X^*. Let d be a $\sigma(X, F)$ lower semicontinuous metric on X generating a topology finer than $\sigma(X, F)$, and such that the identity map is norm to d uniformly continuous on bounded sets of X. Let us assume we also have:

$$d - diam(\mathrm{co}(A)) \leq K_A \cdot d - diam(A)$$

for certain $K_A > 0$ and every bounded subset A of X.
*If there is a sequence of sets (A_n) in X such that for every $x \in X$ and every $\epsilon > 0$ there is $p \in N$ and a $\sigma(X, F)$-open half space H such that $x \in H \cap A_p$ is not empty and can be covered by finitely many sets of d-diameter less than ϵ, then X admits an equivalent $\sigma(E, F)$-lower semicontinuous and d-**LUR** norm, in particular $\sigma(X, F)$-**LUR** norm.*

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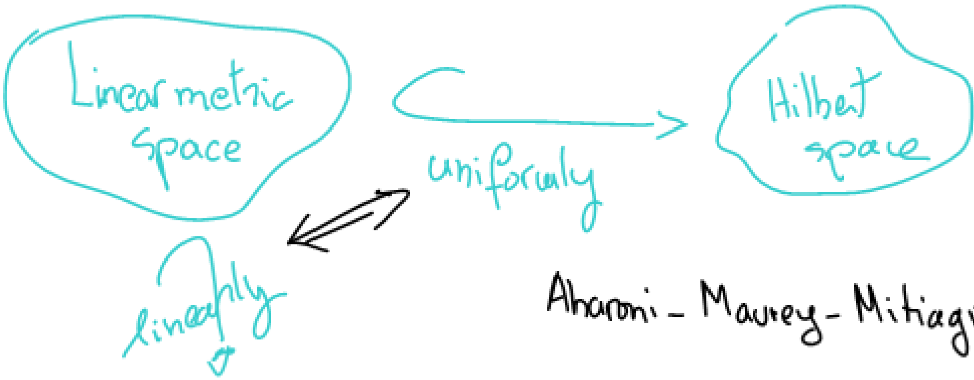
Metrizable topological vector spaces

S. Banach "Theory of linear operations"
V. L. Klee
G. Köthe
H. Jarchow
N. Kalton



Aharoni - Maurey - Mitiagin

$$\mathbb{L}^0(\Omega,\mathfrak{F},\mathbb{P}) \longrightarrow \mathbb{R}^+$$

$$X \longmapsto \int_\Omega \frac{|X(\omega)|}{1+|X(\omega)|}\, d\mathbb{P}(\omega)$$

Is there an equivalent d translation invariant metric on $\mathbb{L}^0(\Omega,\mathfrak{F},\mathbb{P})$ such that $|X| := d(X,0)$ verifies

$$\lim_{n\to\infty}\left[2|X_n|^2 + 2|X|^2 - |X+X_n|^2\right] = 0 \Rightarrow (X_n) \xrightarrow{\mathbb{P}} X \, ?$$

(F)-norm

Definition

A function

$$\|\cdot\| : X \longrightarrow [0, +\infty)$$

is called (F)-norm on the vector space X if the following properties are satisfied:

- $x = 0$ *if, and only if,* $\|x\| = 0$;
- $\|\lambda x\| \leq \|x\|$, *if* $|\lambda| \leq 1$ *and* $x \in X$;
- $\|x + y\| \leq \|x\| + \|y\|$ *for every* $x, y \in X$;
- $\lim_n \|\lambda x_n\| = 0$, *if* $\lim_n \|x_n\| = 0$ *for every* $(x_n)_{n\in\mathbb{N}} \subseteq X$ *and* $\lambda \in \mathbb{R}$;
- $\lim_n \|\lambda_n x\| = 0$, *if* $\lim_n \lambda_n = 0$ *for every* $(\lambda_n)_{n\in\mathbb{N}}$ *and* $x \in X$.

Kadec $\Rightarrow$ LUR renormings

Theorem (S. Ferrari, L. Oncina, M. Raja and J.O)

If a normed space $(X, \|\cdot\|)$ has a Kadec norm there is an equivalent Kadec and locally uniformly rotund (F)-norm $\|\cdot\|_1$ on X, i.e. an (F)-norm $\|\cdot\|_1$ such that the topology determined by the (F)-norm $\|\cdot\|_1$ on X coincides with the norm topology and moreover we have:

1. *the weak and norm topologies coincide on every sphere $\{x \in X : \|x\|_= \rho\}$ for $\rho > 0$.*
2. *For every $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$ we have $\lim_{n\to\infty} \|x_n - x\| = 0$ whenever*

$$\lim_{n\to\infty} (2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2) = 0$$

Examples

R. Haydon
J. Jayne
I. Namioka
C.A. Rogers

The lexicographic product $K = [0,1]^{\omega_1}$ gives

$C(K)$ with a Kadec renorming

but $C(K)$ does not have equivalent LUR norm

"Continuous functions on totally ordered spaces that are compact in their order topologies" J. Funct. Anal. 178, 23-63 (2000)

A. Moltó
S. Troyanski
M. Valdivia
J. O.

If X is a Banach space with a Kadec norm and the Krein-Milman property then X has an equivalent LUR norm

"Kadec and Krein-Milman properties" C.R. Acad. Sci. Paris 131, Série I, 459-464 (2000)

Kadec (F)-renorming $\Leftrightarrow$ descriptiveness

Theorem (S. Ferrari, L. Oncina, M. Raja and J.O.)

Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*. TFAE:

1. *There is a norm-equivalent and $\sigma(X,Z)$-lower semicontinuous (F)-norm $\|\cdot\|_0$ on X such that $\sigma(X,Z)$ and norm topologies coincide on the unit sphere*

$$\{x \in X : \|x\|_0 = 1\}$$

2. *There are isolated families $\mathcal{B}_n$ for the $\sigma(X,Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in \mathcal{B}_n$ with the property that $x \in B$ and*

$$\|\cdot\| - \mathrm{diam}(B) < \epsilon$$

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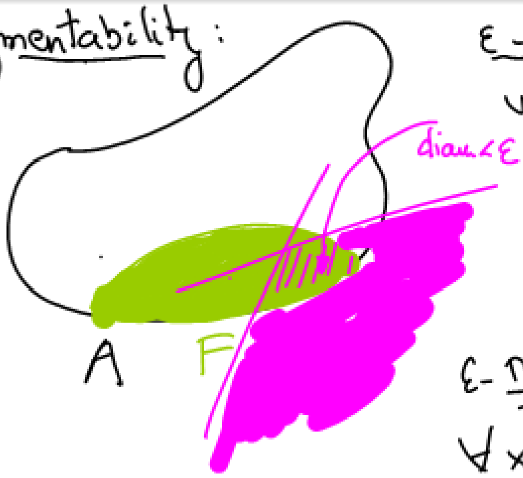
Descriptive Banach spaces

Fragmentability:



ε-fragmented if $\forall F \subset A$ $\exists W$ w-open, $W \cap F \neq \emptyset$ and $\|\cdot\|\text{-diam}(W \cap F) < \varepsilon$

ε-σ-fragmented $A = \bigcup_{n=1}^{\infty} A_{n,\varepsilon}$ s.t. $A_{n,\varepsilon}$ is ε-fragmented $n = 1, 2, \dots$

ε-DESCRIPTIVE: $A = \bigcup_{n=1}^{\infty} A_{n,\varepsilon}$ s.t. $\forall x \in A_{n,\varepsilon}$ $\exists W$ w-open, $x \in W$ and $\|\cdot\|\text{-diam}(W \cap A_{n,\varepsilon}) < \varepsilon$

X DESCRIPTIVE $\Longleftrightarrow$ There are families $\mathcal{B}_n$, relatively discrete for the weak topology, (isolated families) s.t. $\forall x \in X$, $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$, $x \in B \in \mathcal{B}_{n_0}$, $\|\cdot\|\text{-diam}(B_{n_0}) < \varepsilon$

Kadec norm $\Rightarrow$ descriptive $\Rightarrow$ σ-fragmented

w*-descriptive $\Longleftrightarrow$ dual LUR (M. Raja)

Kadec meets Bing-Nagata-Smirnov-Stone

Theorem (Kadec metrization)

Let $(X, \|\cdot\|)$ be a normed space with a norming subspace Z in X^*. Then the following conditions are equivalent:

1. *The normed space X is $\sigma(X, Z)-$descriptive; i.e there are isolated families $\mathcal{B}_n$ for the $\sigma(X, Z)$-topology, $n = 1, 2, \cdots$ such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in \mathcal{B}_n$ with the property that $x \in B$ and $\|\cdot\| - \text{diam}(B) < \epsilon$*
2. *The norm topology admits a basis $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ such that each one of the families $\mathcal{B}_n$ is norm discrete and $\sigma(X, Z)$-isolated*

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2. *There is a norm-equivalent, $\sigma(X,Z)$-lower semicontinuous Kadec and LUR (F)-norm $\|\cdot\|_0$ on X; i.e. such that the weak and the norm topologies coincide on spheres $\{x \in X : \|x\|_0 = \theta\}$ and, for every $(x_n)_{n\in\mathbb{N}} \subset X$ and $x \in X$ we have $\lim_{n\to+\infty} \|x_n - x\| = 0$ whenever*

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Method of proof

Definition (*p*-convex set and hull)

Let A be a subset of a vector space X and $p \in (0,1]$. A is said to be p-convex if for every $x, y \in A$ and $\tau, \mu \in [0,1]$ such that $\tau^p + \mu^p = 1$ we have $\tau x + \mu y \in A$.

If A is p-convex and absorbent, its p-Minkowski functional is

$$p_A(x) := \inf\{\lambda^p : \lambda > 0, x \in \lambda A\}$$

p_A is a p-seminorm, i.e we have

- $p_A(\lambda x) = |\lambda|^p p_A(x)$
- $p_A(x+y) \leq p_A(x) + p_A(y)$.

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p-convex functions

Definition

A real function $\phi : X \longrightarrow \mathbb{R}$ *is said to be p-convex for* $p \in (0,1]$ *if*

$$\phi(\tau x + \mu y) \leq \tau\phi(x) + \mu\phi(y)$$

whenever $\tau \geq 0$, $\mu \geq 0$ *and* $\tau^p + \mu^p = 1$.

- the epigraph of ϕ is p-convex if and only if ϕ is p-convex;
- if ϕ is convex and $\phi(0) = 0$, then ϕ is p-convex for every $p \in (0,1]$;
- if ϕ_p is p-convex, ϕ_q is q-convex, with $0 < p \leq q < 1$ and both of them are non-negative, then $\phi_p + \phi_q$ is p-convex;
- if $\phi : X \to \mathbb{R}$ is p-convex for some $0 < p \leq 1$ and bounded from above in a neighbourhood of $x \in X$, then ϕ is locally Lipschitz at x
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p-convex functions

Definition (p-distance)

Let C be a w*-compact and p-convex subset of X^{**}, $0 < p \leq 1$,

$$\varphi(x) := \inf_{c^{**} \in C} \{\sup\{|\langle x - c^{**}, z^* \rangle| : z^* \in B_{X^*} \cap Z\}\}$$

φ is a p-convex, $\sigma(X, Z)$-lower semicontinuous and 1-Lipschitz map from X to $[0, +\infty)$.

Definition

A family $\mathcal{B} := \{B_i : i \in I\}$ of subsets in the normed space X is said to be p-isolated for the $\sigma(X, Z)$-topology if for every $i \in I$

$$B_i \cap \overline{\mathrm{co}_p\{B_j : j \neq i,\ j \in I\}}^{\sigma(X,Z)} = \emptyset.$$

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Orthogonal p-convex sets and functions

Theorem

Let $\mathcal{B} := \{B_i \; i \in I\}$ be an uniformly bounded family of subsets of X. The following are equivalent:

1. *The family $\mathcal{B}$ is p-isolated for the $\sigma(X,Z)$-topology; i.e.*

$$B_i \cap \overline{\mathrm{co}_p\{B_j : j \neq i, \; j \in I\}}^{\sigma(X,Z)} = \emptyset.$$

   for every $i \in I$

2. *There exists a family $\mathcal{L} := \{\varphi_i : X \to [0, +\infty) : i \in I\}$ of p-convex and $\sigma(X,Z)$-lower semicontinuous functions such that for every $i \in I$*

$$\{x \in X : \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i.$$

Orthogonal p-convex sets and functions

Theorem

Let $\mathcal{B} := \{B_i \; i \in I\}$ be an uniformly bounded family of subsets of X. The following are equivalent:

1. *The family $\mathcal{B}$ is p-isolated for the $\sigma(X, Z)$-topology; i.e.*

$$B_i \cap \overline{\mathrm{co}_p\{B_j : j \neq i, \; j \in I\}}^{\sigma(X,Z)} = \emptyset.$$

for every $i \in I$

2. *There exists a family $\mathcal{L} := \{\varphi_i : X \to [0, +\infty) : i \in I\}$ of p-convex and $\sigma(X, Z)$-lower semicontinuous functions such that for every $i \in I$*

$$\{x \in X : \varphi_i(x) > 0\} \cap \bigcup_{j \in I} B_j = B_i.$$

Isolatedness $\Rightarrow$ *p*-convex isolatedly decomposable

Lemma (Decomposition lemma)

Let $\mathcal{B}$ be a uniformly bounded and isolated family of sets for the $\sigma(X,Z)$ topology. Then for every $B \in \mathcal{B}$ we can write

$$B = \bigcup_{n=1}^{\infty} B_n$$

in such a way that, for every $n \in \mathbb{N}$ fixed, the family

$$\{B_n : B \in \mathcal{B}\}$$

is $\sigma(X,Z)$-q-isolated whenever $q < \frac{\log 2}{\log 4n}$.

p-Localization Theorem

Theorem (p=1 S. Troyanski and J.O; 0<p<1 S. Ferrari, L. Oncina, M. Raja and J.O.)

Let $\mathcal{B} = \{B_i : i \in I\}$ be a uniformly bounded and p-isolated family of subsets of X for the $\sigma(X, Z)$ topology. Then there is a norm-equivalent $\sigma(X, Z)$-lower semicontinuous p-norm $q_{\mathcal{B}}(\cdot)$ on X such that for every $i_0 \in I$, every $x \in B_{i_0}$, and every sequence $(x_n)_{n\in\mathbb{N}}$ in X the condition

$$\lim_{n\to+\infty} [2q^2_{\mathcal{B}}(x_n) + 2q^2_{\mathcal{B}}(x) - q^2_{\mathcal{B}}(x + x_n)] = 0,$$

implies that:

1. *there exists $n_0 \in \mathbb{N}$ such that $x_n, \frac{x_n+x}{2^{1/p}} \notin \overline{\mathrm{co}_p\{B_i \ : i \neq i_0, \ i \in I\}}^{\sigma(X,Z)}$ for every $n \geq n_0$;*
2. *for every positive δ there is $n_\delta \in \mathbb{N}$ such that $x_n \in \overline{\mathrm{co}(B_{i_0} \cup \{0\}) + B(0, \delta)}^{\sigma(X,Z)}$ whenever $n \geq n_\delta$.*

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Descriptive $\Rightarrow$ LUR (F)-renorming

- Fix isolated families $\mathcal{B}_n$ for the $\sigma(X,Z)$-topology such that for every $x \in X$ and every $\epsilon > 0$ there is $n \in \mathbb{N}$ and some set $B \in \mathcal{B}_n$ with $x \in B$ and $\|\cdot\| - \operatorname{diam}(B) < \epsilon$.
- $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ are assumed to be p_n-isolated for some sequence $p_n \in (0,1]$ by decomposition lemma.
- Consider the p_n-norms $q_{\mathcal{B}_n}(\cdot)$ constructed using the p-Localization Theorem
- $F^2_{\mathcal{B}}(x) := \|x\|^2_Z + \sum_{n=1}^{+\infty} \frac{1}{\zeta_n^{2p_n} 2^n} q^2_{\mathcal{B}_n}(x)$ where $q_{\mathcal{B}_n}(x) \leq \zeta_n^{p_n}\|x\|^{p_n} \leq \zeta_n^{p_n} \max\{1, \|x\|\}$.
- If $\lim_{n\to+\infty}[2F^2_{\mathcal{B}}(x_n) + 2F^2_{\mathcal{B}}(x) - F^2_{\mathcal{B}}(x + x_n)] = 0$ then $\lim_{n\to+\infty}[2q^2_{\mathcal{B}_m}(x_n) + 2q^2_{\mathcal{B}_m}(x) - q^2_{\mathcal{B}_m}(x + x_n)] = 0$ for all m.
- If $\epsilon > 0$, $m \in \mathbb{N}$ and $B_0 \in \mathcal{B}_m$ with $x \in B_0 \subseteq x + \frac{\epsilon}{2} B_X$ there exists $n_{\frac{\epsilon}{2}}$ such that $x_n \in \overline{\operatorname{co}(B_0 \cup \{0\}) + B(0, \frac{\epsilon}{2})}^{\sigma(X,Z)}$ whenever $n \geq n_{\frac{\epsilon}{2}}$.

Descriptive $\Rightarrow$ LUR (F)-renorming

- $\|\cdot\|\text{dist}(x_n, l_x) \leq \epsilon$ for $n \geq n_{\frac{\epsilon}{2}}$
- there is $r_{(n,\epsilon)} \in [0,1]$ such that $\|x_n - r_{(n,\epsilon)}x\| \leq \epsilon$ for $n \geq n_{\frac{\epsilon}{2}}$.
- By induction we select integers $n_1 < n_2 < \cdots < n_k < \cdots$ such that $\|x_{n_k} - r_{(n_k,1/k)}x\| \leq \frac{1}{k}$.
- By compactness there is a sequence of integers $k_1 < k_2 < \cdots < k_j < \cdots$ such that $\lim_{j\to+\infty} r_{(n_{k_j},1/k_j)} = r \in [0,1]$ and $\|\cdot\| - \lim_{j\to+\infty} x_{n_{k_j}} = rx$
- If $\|x\|_Z = 1$ we also have $\lim_n \|x_n\|_Z = \|x\|_Z = 1$ and it follows that $r = 1$, so we have found a subsequence (x_{n_j}) of the given sequence (x_n) which norm converges to x
- Since the reasoning is valid for every subsequence too, the proof is over

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Haydon's lemma + Burke, Kubis and Todorcevic

Lemma

Let X be a topological space, S be a set and $\varphi_s, \psi_s : X \to [0, +\infty)$ lower semicontinuous functions such that $\sup_{s\in S}(\varphi_s(x) + \psi_s(x)) < +\infty$ for every $x \in X$. Define

$$\varphi(x) = \sup_{s\in S} \varphi_s(x), \quad \theta_m(x) = \sup_{s\in S}(\varphi_s(x) + 2^{-m}\psi_s(x)),$$

and $\theta(x) = \sum_{m\in\mathbb{N}} 2^{-m}\theta_m(x)$. Assume further that $\{x_\sigma : \sigma \in \Sigma\}$ is a net converging to $x \in X$ and $\theta(x_\sigma) \to \theta(x)$. Then there exists a finer net $\{x_\gamma\}_{\gamma\in\Gamma}$ and a net $\{i_\gamma\}_{\gamma\in\Gamma} \subseteq S$ such that

$$\lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x_\gamma) = \lim_{\gamma\in\Gamma} \varphi_{i_\gamma}(x) = \lim_{\gamma\in\Gamma} \varphi(x_\gamma) = \sup_{s\in S} \varphi_s(x)$$

and

$$\lim_{\gamma\in\Gamma}(\psi_{i_\gamma}(x_\gamma) - \psi_{i_\gamma}(x)) = 0.$$

p-connection with R. Haydon

Theorem

Let $\mathcal{B} := \{B_i : i \in I\}$ be an uniformly bounded and p-isolated family of subsets of X for the $\sigma(X,Z)$-topology and some $p \in (0,1]$. Then there is an equivalent $\sigma(X,Z)$-lower semicontinuous quasinorm, with p-power a p-norm, $\|\cdot\|_{\mathcal{B}}$ on X such that: for every net $\{x_\alpha : \alpha \in A\}$ and x in X with $x \in B_{i_0}$ for $i_0 \in I$, the conditions $\sigma(X,Z) - \lim_\alpha x_\alpha = x$ and $\lim_\alpha \|x_\alpha\|_{\mathcal{B}} = \|x\|_{\mathcal{B}}$ imply that

1. *there exists $\alpha_0 \in A$ such that x_α is not in $\overline{\mathrm{co}_p\{B_i \; i \neq i_0, \; i \in I\}}^{\sigma(X,Z)}$ for $\alpha \geq \alpha_0$;*
2. *for every positive δ there exists $\alpha_\delta \in A$ such that*

$$x, x_\alpha \in \overline{\mathrm{co}(B_{i_0} \cup \{0\}) + B(0,\delta)}^{\sigma(X,Z)}$$

   whenever $\alpha \geq \alpha_\delta$.

Descriptive ⇒ LUR + Kadec (F)-renorming

- We can construct norm-equivalent and $\sigma(X, Z)$-lower semicontinuous F-norms F_1 and F_2 such that F_1 has the LUR property and F_2 the Kadec property.
- Then we define

$$\|\cdot\|_1(x)^2 := F_1(\cdot)^2 + F_2(\cdot)^2$$

  which is an equivalent $\sigma(X, Z)$-lower semicontinuous F-norm which has both Kadec and the LUR property.
- $\lim_{n\to\infty}[2\|x\|_1^2 + 2\|x_n\|_1^2 - \|x + x_n\|_1^2] = 0$ is equivalent to $\lim_{n\to\infty}[2F_i(x)^2 + 2F_i(x_n)^2 - F_i(x + x_n)^2] = 0$ for $i = 1, 2$, and LUR property of F_1 is translated to $\|\cdot\|_1$.
- If $\{x_\alpha : \alpha \in (A, \succ)\}$ is a net in X which converges to x in the topology $\sigma(X, Z)$ and $\lim_{\alpha\in A}\|x_\alpha\|_1 = \|x\|_1$ it follows that $\lim_{\alpha\in A} F_i^2(x_\alpha) = F_i^2(x)$ for $i = 1, 2$. Thus Kadec property of F_2 is translated to $\|\cdot\|_1$.

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José Martí Poem for Robert Deville

Cultivo la rosa blanca
tanto en julio como en enero,
para el amigo sincero
que me da su mano franca.
Y para el cruel que arranca
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