The notion of support for elements of Lipschitz-free spaces

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Let (M, d) be a complete metric space with a distinguished point $0 \in M$ (called a *pointed metric space*).

Then

$$\mathrm{Lip}_0(M) = \{f : M \longrightarrow \mathbb{R} \ : \ f \text{ Lipschitz}, \ f(0) = 0\}$$

with

$$\|f\|_L = \mathrm{Lip}(f) = \sup\left\{\frac{|f(p) - f(q)|}{d(p,q)} \ : \ p, q \in M, \ p \neq q\right\}$$

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Moreover, if M is bounded, then $\mathrm{Lip}_0(M)$ is an algebra with pointwise multiplication:

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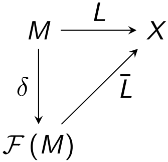
- 

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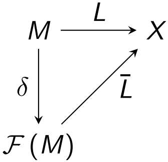
- 

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- $\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$ and $f_i \xrightarrow{w^*} f \iff f_i \xrightarrow{p.w.} f$ in $B_{\mathrm{Lip}_0(M)}$.

Theorem (Aliaga, P., '19)

Let M be a complete pointed metric space and let $\{K_i : i \in I\}$ be a family of closed subsets of M containing the base point. Then

$$\bigcap_{i\in I} \mathcal{F}(K_i) = \mathcal{F}\left(\bigcap_{i\in I} K_i\right).$$

If $K \subseteq M$, then $\mathcal{F}(K \cup \{0\}) \equiv \overline{\mathrm{span}}\, \delta(K) \subseteq \mathcal{F}(M)$.

For a closed $K \subseteq M$, define the *kernel* of K as

$$\mathcal{I}(K) = \{f \in \mathrm{Lip}_0(M) : \ f(p) = 0, \ \forall p \in K\}.$$

Then $\mathcal{F}(K)^{\perp} = \mathcal{I}(K)$ and $\mathcal{I}(K)_{\perp} = \mathcal{F}(K)$.

Lemma

If M and K_i are as in Theorem, then

$$\overline{\mathrm{span}}^{w^*}\{\mathcal{I}(K_i) : \ i \in I\} = \mathcal{I}\left(\bigcap_{i\in I} K_i\right).$$

Proof of Theorem:

$$\bigcap_{i\in I}\mathcal{F}(K_i) = \bigcap_{i\in I}(\mathcal{I}(K_i)_{\perp}) = \left(\bigcup_{i\in I}\mathcal{I}(K_i)\right)_{\perp} =$$

$$\left(\overline{\mathrm{span}}^{w^*}\{\mathcal{I}(K_i) : \ i \in I\}\right)_{\perp} = \mathcal{I}\left(\bigcap_{i\in I} K_i\right)_{\perp} = \mathcal{F}\left(\bigcap_{i\in I} K_i\right).$$

□

Proof of Lemma for bounded M:

- If $m \in \mathcal{F}(M)$, $g \in \mathrm{Lip}_0(M)$ and we define $(m \circ g)(f) = \langle m, f.g \rangle$ for $f \in \mathrm{Lip}_0(M)$, then $m \circ g \in \mathcal{F}(M)$.
- If Y is an ideal of $\mathrm{Lip}_0(M)$, then $\overline{Y}^{w^*}$ is also an ideal of $\mathrm{Lip}_0(M)$.
- For $K \subseteq M$, the kernel $\mathcal{I}(K)$ is a w^*-closed ideal of $\mathrm{Lip}_0(M)$.

Theorem (Weaver, '95)

If $\mathcal{A}$ is a w^*-closed ideal of $\mathrm{Lip}_0(M)$, then $\mathcal{A} = \mathcal{I}(\mathcal{H}(\mathcal{A}))$, where

$$\mathcal{H}(\mathcal{A}) = \{p \in M : \ f(p) = 0, \ \forall f \in \mathcal{A}\}.$$

- Finally,

$$\overline{\mathrm{span}}^{w^*}\{\mathcal{I}(K_i) : \ i \in I\} = \mathcal{I}\left(\mathcal{H}\left(\overline{\mathrm{span}}^{w^*}\{\mathcal{I}(K_i) : \ i \in I\}\right)\right) = \mathcal{I}\left(\bigcap_{i \in I} K_i\right).$$

Extension to unbounded M:

- Let $f \in \mathcal{I}\left(\bigcap_{i\in I} K_i\right)$ and let U be a w^*-neighbourhood of f. We want to show that
$$\operatorname{span}\{\mathcal{I}(K_i) : i \in I\} \cap U \neq \emptyset.$$
- Lipschitz functions with bounded supports are w^*-dense in $\mathrm{Lip}_0(M)$ and in $\mathcal{I}(K)$, so we may assume that f has a bounded support.
- Let $A \subseteq M$ be a bounded set containing the base point and let $g \in \mathrm{Lip}(M)$ with $\operatorname{supp}(g) \subseteq A$. For a $\kappa \in \mathrm{Lip}_0(A)$ define
$$T_g(\kappa)(p) = \begin{cases} g(p)\kappa(p) & \text{if } p \in A \\ 0 & \text{if } p \notin A \end{cases}.$$
Then $T_g : \mathrm{Lip}_0(A) \to \mathrm{Lip}_0(M)$, $T_g(\mathcal{I}(K \cap A)) \subseteq \mathcal{I}(K)$ and T_g is w^*-w^*-continuous. Hence $(T_g)_* : \mathcal{F}(M) \to \mathcal{F}(A)$. In particular, if M is bounded and $A = M$, then $(T_g)_*(m) = m \circ g$. □

Definition

Let M be a complete pointed metric space. For an $m \in \mathcal{F}(M)$, we define the **support** of m, denoted $\operatorname{supp}(m)$, as the intersection of all closed subsets K of M such that $m \in \overline{\operatorname{span}}\ \delta(K) \equiv \mathcal{F}(K \cup \{0\})$.

Proposition

The support of m is the smallest closed set $K \subseteq M$ such that $m \in \mathcal{F}(K \cup \{0\})$.

Proof: The intersection theorem. □

Properties of supports:

- For $m \in \mathcal{F}(M)$ and $f, g \in \mathrm{Lip}_0(M)$,

$$f|_{\mathrm{supp}(m)} = g|_{\mathrm{supp}(m)} \implies \langle m, f\rangle = \langle m, g\rangle.$$

- For $m \in \mathcal{F}(M)$ and $p \in M$,

$$p \in \mathrm{supp}(m) \iff \forall \text{ neighbourhood } U \text{ of } p \ \exists\, f \in \mathrm{Lip}_0(M) \text{ s. t.}$$

$$\mathrm{supp}(f) \subseteq U \text{ and } \langle m, f\rangle > 0.$$

- Theorem (Aliaga, '19)

  Let M be a complete pointed metric space and let $m \in \mathcal{F}(M)$. Suppose that $\mathrm{supp}(m) \subseteq S_1 \cup S_2$, *where $S_1, S_2 \subseteq M$ are closed, $d(S_1, S_2) > 0$ and S_1 is bounded. Then there exists a unique decomposition*

$$m = m_1 + m_2,$$

  where $m_1, m_2 \in \mathcal{F}(M)$ are such that $\mathrm{supp}(m_1) \subset S_1$ *and* $\mathrm{supp}(m_2) \subset S_2$.

  If m is positive, then it is enough if S_1, S_2 are closed and disjoint. In fact, for a positive m and a closed set $S \subseteq M$ there exists a *restriction of m to S*.

Let μ be a Radon measure on M. Then

$$\int_M \delta(p)\, d\mu(p) \in \mathcal{F}(M) \iff d(\cdot, 0) \in L_1(|\mu|)$$

and

$$\left\langle \int_M \delta(p)\, d\mu(p), f \right\rangle = \int_M f(p)\, d\mu(p).$$

Proposition

If $m \in \mathcal{F}(M)$ is induced by a Radon measure μ on M, then the support of m agrees with the support of μ, possibly up to the base point.

Proposition

If $m \in \mathcal{F}(M)$ is induced by a Radon measure on M, then there exists $C > 0$ such that

$$m \in \overline{\left\{ \sum_{i=1}^{n} a_i \delta(p_i) \in \mathcal{F}(M) : \sum_{i=1}^{n} |a_i| \leq C \right\}}.$$

If M is locally compact then the converse also holds.

Theorem (Aliaga, P., '19)

Let M be a complete pointed metric space and let $m \in \mathcal{F}(M)$ be such that $0 \notin \operatorname{supp}(m)$. TFAE:

1. *there exists a positive $\nu \in \mathcal{F}(M)$ such that $m \leq \nu$ (i.e. $\nu - m$ is positive),*
2. *there exist positive $m^+, m^- \in \mathcal{F}(M)$ such that $m = m^+ - m^-$,*
3. *m is induced by a Radon measure on M.*

Thank you for your attention!

References

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