Weak sequential completeness of Lipschitz-free spaces

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June 21, 2017

Definition (Lipschitz-free space)

Let (M, d) be a metric space with a distinguished element $0 \in M$. Then

$$\mathrm{Lip}_0(M) = \{f : M \longrightarrow \mathbb{R} \ : \ f \text{ Lipschitz}, \ f(0) = 0\}$$

with

$$\|f\| = \mathrm{Lip}(f) = \sup\left\{\frac{|f(p) - f(q)|}{d(p, q)} \ : \ p, q \in M, \ p \neq q\right\}$$

is a Banach space. The **Lipschitz-free space** over M is the space

$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|}\{\delta_M(p) \ : \ p \in M\} \subseteq \mathrm{Lip}_0(M)^*,$$

where $\langle f, \delta_M(p)\rangle = f(p)$ for $f \in \mathrm{Lip}_0(M)$ and $p \in M$.

$\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$

Definition (Weak sequential completeness)

A sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X is **weakly Cauchy** if the sequence $(\langle x_n, x^*\rangle)_{n=1}^{\infty}$ is convergent for every $x^* \in X^*$. A Banach space X is called **weakly sequentially complete** if every weakly Cauchy sequence in X is weakly convergent.

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Theorem (Cúth, Doucha, Wojtaszczyk, 2016)

The Lipschitz-free space $\mathcal{F}([0,1]^n)$ is weakly sequentially complete.

Equivalent: $\mathcal{F}(\mathbb{R}^n)$, $\mathcal{F}(M)$ for any $M \subset \mathbb{R}^n$

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If M is a compact subset of a superreflexive Banach space, then the Lipschitz-free space $\mathcal{F}(M)$ is weakly sequentially complete.

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Examples

- For each $p \in (1,\infty)$, let $\mathcal{Q}_p = \prod_{n=1}^{\infty}[0, \frac{1}{n}]$ be the Hilbert cube in ℓ_p. $\mathcal{F}(\mathcal{Q}_p)$ is weakly sequentially complete.
- Lafforgue, Naor (2014): For each $p \in (2,\infty)$ there exists a noncompact doubling subset $\mathcal{M}_p$ of L_p such that $\mathcal{M}_p \overset{\text{bi-Lip}}{\not\hookrightarrow} L_q$ for any $q \in (1,p)$.
  Kalton (2004): $\mathcal{F}(\mathcal{M}_p) \hookrightarrow \left(\bigoplus_{k=1}^{\infty} \mathcal{F}(\mathcal{M}_{p,k})\right)_{\ell_1}$, where
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Theorem (Bourgain, 1983)

The space $(C^1([0,1]^n))^*$ *is weakly sequentially complete.*

Proof

1. $C^1([0,1]^n) \overset{\Phi}{\hookrightarrow} C([0,1]^n, \ell_\infty^{n+1})$ by $\Phi(F) = (F, \partial_1 F, \ldots, \partial_n F)$.
2. Let $\Gamma \subset (\Phi(C^1([0,1]^n)))^*$ be bounded and not relatively weakly compact. Find $\xi > 0$, $(f_1^k, \ldots, f_k^k) \subset \Phi(C^1([0,1]^n))$, $\|f_i^k\| \leq 1$, and $(\mu_1^k, \ldots, \mu_k^k) \subset \Gamma$ such that $\left|\left\langle f_i^k, \mu_j^k \right\rangle\right| \leq \frac{\xi}{3}$ for $j < i$ and $\left\langle f_i^k, \mu_j^k \right\rangle \geq \xi$ for $i \leq j$.
3. If H is a Hilbert space, $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in B_H$, then there exist sets $A, B \subset \{1, \ldots, k\}$ such that $\max A < \min B$ and $\left\| \frac{1}{|A|} \sum_{i \in A} x_i - \frac{1}{|B|} \sum_{i \in B} x_i \right\| \leq \frac{4}{\sqrt{\log k}}$.
4. Apply (3) to $f_1^k, \ldots, f_k^k \in L_2([0,1]^n, |\mu_j^m| + \lambda, \ell_\infty^{n+1})$ for $k < m$. Build thus inductively sequences $(z_i)_{i=1}^\infty \subset \Phi(C^1([0,1]^n))$, $\|z_i\| \leq 1$, $(\nu_i)_{i=1}^\infty \subset \Gamma$ s. t.
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Thank you for your attention!