Approximation properties in Lipschitz-free spaces

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Definition (Lipschitz-free space)

Let (M, d) be a metric space with a distinguished element $0 \in M$. Then

$$\mathrm{Lip}_0(M) = \{f : M \longrightarrow \mathbb{R} \ : \ f \text{ Lipschitz}, \ f(0) = 0\}$$

with

$$\|f\| = \mathrm{Lip}(f) = \sup\left\{\frac{|f(p) - f(q)|}{d(p, q)} \ : \ p, q \in M, \ p \neq q\right\}$$

is a Banach space. The **Lipschitz-free space** over M is the space

$$\mathcal{F}(M) = \overline{\mathrm{span}}^{\|\cdot\|}\{\delta_M(p) \ : \ p \in M\} \subseteq \mathrm{Lip}_0(M)^*,$$

where $\langle f, \delta_M(p)\rangle = f(p)$ for $f \in \mathrm{Lip}_0(M)$ and $p \in M$.

$\mathcal{F}(M)^* \equiv \mathrm{Lip}_0(M)$

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Definition (BAP)

A Banach space X has

- the **approximation property (AP)** if, given $K \subseteq X$ compact and $\varepsilon > 0$, there is a bounded finite-rank operator T on X such that $\|Tx - x\| \leq \varepsilon$ for all $x \in K$;
- the $\lambda-$**bounded approximation property (**$\lambda-$**BAP)**, $1 \leq \lambda < \infty$, if, moreover, $\|T\| \leq \lambda$;
- the **metric approximation property (MAP)** if it has $1-$BAP.

Theorems (Godefroy, Kalton, 2003)

If X is a finite-dimensional Banach space then $\mathcal{F}(X)$ has the MAP.
A Banach space X has the λ-BAP if and only if $\mathcal{F}(X)$ has the λ-BAP.

Theorem (Godefroy, Ozawa, 2014)

There exists a compact metric space K such that $\mathcal{F}(K)$ fails AP.

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Problem

For which metric spaces M does $\mathcal{F}(M)$ have the BAP?

Theorem (Lancien, P, 2013)

If M is a doubling metric space (i.e. there exists $D(M) > 0$ such that any open ball with radius R can be covered with $D(M)$ open balls of radius $\frac{R}{2}$), then $\mathcal{F}(M)$ has the $C(1 + \log(D(M)))$-BAP, where C is a universal constant. In particular, if $M \subseteq \ell_2^N$ then $\mathcal{F}(M)$ has CN-BAP.

Proposition (Lancien, P, 2013)

If $M \subseteq \ell_2^N$ then $\mathcal{F}(M)$ has $C\sqrt{N}$-BAP.

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Is there $\lambda > 0$ such that for every $N \in \mathbb{N}$ and $M \subseteq \mathbb{R}^N$, $\mathcal{F}(M)$ has the $\lambda-$BAP?

Theorem (P, Smith, 2014)

If $M \subseteq (\mathbb{R}^N, \|\cdot\|)$ is a compact set such that the set

$$\{x \in M \ : \ [x, y] \subseteq M \text{ for all } y \in M\}$$

has non-empty interior, then $\mathcal{F}(M)$ has the MAP.

Corollary (P, Smith, 2014)

If K is a compact and convex subset of a finite-dimensional Banach space, then $\mathcal{F}(K)$ has the MAP.

Theorem (Dalet, 2014)

If M is a countable proper metric space or a proper ultrametric space, then $\mathcal{F}(M)$ has the MAP.

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Let X be a Banach space. A sequence $(X_n)_{n=1}^{\infty}$ of finite-dimensional subspaces of X is called a **finite-dimensional Schauder decomposition (FDD)** of X if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for every $n \in \mathbb{N}$. The **decomposition constant** is defined as $\sup_{k \in \mathbb{N}} \|S_k\|$, where $S_k\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{k} x_n$.

Theorem (Borel-Mathurin, 2012)

$\mathcal{F}(\mathbb{R}^N)$ has an FDD. The decomposition constant depends on the dimension N.

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$\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_1^N)$ have a monotone FDD, i.e. with decomposition constant 1.

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Theorem (Hájek, P, 2014)

$\mathcal{F}(\ell_1)$ and $\mathcal{F}(\ell_1^N)$ have a Schauder basis.

Problem

Does $\mathcal{F}(M)$ have a Schauder basis for every $M \subseteq \mathbb{R}^N$?

Theorem (Kaufmann, 2014)

If $M \subseteq \mathbb{R}^N$ has non-empty interior, then $\mathcal{F}(M)$ is isomorphic to $\mathcal{F}(\mathbb{R}^N)$.

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Thank you for your attention!

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