On the embeddability of the family of countably Branching trees into quasi-reflexive Banach spaces

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1 Introduction

2 Szlenk index and countably branching trees

3 Asymptotic uniform properties of Banach spaces

$(M,d),(N,\rho)$ metric spaces

- M *Lipschitz embeds* into N : $\exists f : M \to N,\ \exists a,b>0$:

$$\forall x,y \in M,\ \ ad(x,y) \leq \rho(f(x),f(y)) \leq bd(x,y)$$

$(M_i,d_i)_{i\in I}$ family of metric spaces

- $(M_i,d_i)_{i\in I}$ *equi-Lipschitz embeds* into N :$\exists (f_i : M_i \to N)_{i\in I},\ \exists a,b>0$:

$$\forall i \in I,\ \forall x,y \in M_i,\ \ ad_i(x,y) \leq \rho(f_i(x),f_i(y)) \leq bd_i(x,y)$$

Ribe, 1976

Let X, Y be Banach spaces. If X coarse-Lipschitz embeds into Y, then X is (crudely) finitely representable in Y.

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Bourgain, 1986

Let X be a Banach space. Then X is super-reflexive if and only if the family $(D_N)_{N\geq 1}$ of dyadic hyperbolic trees does not equi-Lipschitz embed into X.

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X Banach space, K weak*-compact subset of X^*, $\varepsilon > 0$
$\mathcal{V}$ the set of all weak*-open subsets V of K satisfying diam $V \leq \varepsilon$
$s_\varepsilon(K) = K \setminus \left(\bigcup_{V \in \mathcal{V}} V\right)$

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$S_Z(K,\varepsilon) = \inf \{\alpha \ : \ s_\varepsilon^\alpha(K) = \emptyset\}$ if such an α exists and $= \infty$ otherwise

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Slzlenk index of ℓ_p and c_0

$S_Z(\ell_1) = \infty$
$\forall p \in (1,\infty)$, $S_Z(\ell_p) = \omega$
$S_Z(c_0) = \omega$

$T_N = \{\emptyset\} \cup \bigcup_{n=1}^{N} \mathbb{N}^n$

$s \leq t$: the sequence t is an extension of the sequence s
$a_{s,t}$: greatest common ancestor

$|t|$: length of the sequence t
$d(s,t) = d(a_{s,t}, s) + d(a_{s,t}, t) = |s| + |t| - 2\,|a_{s,t}|$

Reference examples

The family $(T_N)_{N\geq 1}$ equi-Lipschitz embeds into ℓ_1.
The family $(T_N)_{N\geq 1}$ equi-Lipschitz embeds into c_0.

Baudier, Kalton, Lancien

Let X be a Banach space. If $S_Z(X) > \omega$ or if $S_Z(X^*) > \omega$, then $(T_N)_{N\geq 1}$ embeds equi-Lipschitz into X.

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$(X, \|.\|)$ Banach space

- $\|.\|$ AUS : $\exists \rho : [0,\infty) \to [0,\infty),\ \lim_{t\to 0} \frac{\rho(t)}{t} = 0$:

$$\forall x \in S_X,\ \forall t \geq 0,\ \exists V \in \mathcal{V}_w(0) :\ \forall y \in V \cap B_X,\ \|x + ty\| \leq 1 + \rho(t)$$

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- $\|.\|$ AUC : $\exists \delta : [0, \infty) \to [0, \infty),\ \forall t > 0,\ \delta(t) > 0$:

$\forall x \in S_X,\ \forall t > 0,\ \exists V \in \mathcal{V}_w(0) :\ \forall y \in V,\ \|y\| \geq 1,\ \|x + ty\| \geq 1 + \delta(t)$

Knaust, Odell, Schlumprecht

Let X be a separable Banach space. The following assertions are equivalent.

1. The space X admits an equivalent AUS norm.
2. The space X satisfies $S_Z(X) \leq \omega$.

Duality AUC/AUS in reflexive Banach spaces

Let $(X, \|.\|)$ be a reflexive Banach space. Then $\|.\|$ is AUC if and only if $\|.\|_{X^*}$ is AUS.

Baudier, Kalton, Lancien

Let X be a separable reflexive Banach space. If $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$ then $(T_N)_{N\geq 1}$ does not equi-Lipschitz embed into X.

Extension to quasi-reflexive setting, P.

Let X be a quasi-reflexive Banach space. If $S_Z(X) \leq \omega$ and $S_Z(X^*) \leq \omega$ then $(T_N)_{N \geq 1}$ does not equi-Lipschitz embed into X.

(1) weak topology on $X \rightsquigarrow$ weak* topology on X^{**}

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(2) AUC $\rightsquigarrow$ AUC*

$(X, \|.\|)$ Banach space, $|.|$ norm on X^* equivalent to $\|.\|_{X^*}$

- $|.|$ AUC* : $\exists \delta : [0,\infty) \to [0,\infty),\ \forall t > 0,\ \delta(t) > 0$:

$\forall x^* \in S_{X^*},\ \forall t > 0,\ \exists V \in \mathcal{V}_{w^*}(0) :\ \forall y^* \in V,\ |y^*| \geq 1,\ |x^* + ty^*| \geq 1 + \delta(t)$

(1) weak topology on $X \rightsquigarrow$ weak* topology on X^{**}

(2) AUC + duality AUC/AUS $\rightsquigarrow$ AUC* +

Duality AUS/AUC*

Let X be a Banach space. Then $\|.\|_X$ is AUS if and only if $\|.\|_{X^*}$ is AUC*.

(1) weak topology on $X \rightsquigarrow$ weak* topology on X^{**}

(2) AUC + duality AUC/AUS $\rightsquigarrow$ AUC* + duality AUS/AUC*

(3) AUS $\rightsquigarrow$

Lancien, Raja

Let $(X, \|.\|)$ be a AUS Banach space. Then there exists a function $\rho : [0, \infty) \to [0, \infty)$ with $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$ such that

$$\forall x \in S_X,\ \forall t \geq 0,\ \exists V \in \mathcal{V}_\tau(0) :\ \forall y^{**} \in V \cap B_{X^{**}},\ \|x + ty^{**}\| \leq 1 + \rho(t)$$

where τ stands for the weak* topology on X^{**}.

(1) weak topology on $X \rightsquigarrow$ weak* topology on X^{**}

(2) AUC + duality AUC/AUS $\rightsquigarrow$ AUC* + duality AUS/AUC*

(3) AUS + quasi-reflexivity $\rightsquigarrow$ Lancien Raja +

Ramsey

Let (K, ρ) be a compact metric space and let $k \geq 1$. For every map $f : [\mathbb{N}]^k \to K$ and for every $\varepsilon > 0$, there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that diam $f\left([\mathbb{M}]^k\right) \leq \varepsilon$.