The Faces-Radon-Nikodým property in Banach spaces

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First some notions:

1. $\Gamma_0(X) = \{f : X \to \mathbb{R}_\infty : \ f$ proper, convex and lsc$\}$.
2. $\Gamma_0(X^*, w^*) = \{f^* : X^* \to \mathbb{R}_\infty : \ f^* \in \Gamma_0(X^*)$ and w^*-lsc$\}$.
3. For $f \in \Gamma_0(X)$, the (Fenchel) conjugate of f is the function $f^* \in \Gamma_0(X^*, w^*)$ given by

$$f^*(x^*) = \sup_{x \in X}\{\langle x^*, x\rangle - f(x)\}.$$

Remark 1.1

$* : \Gamma_0(X) \to \Gamma_0(X^*, w^*)$ *is a biyection, provided by the equality*

$$f = f^{**}\big|_X.$$

Example 1.2

For a closed convex bounded set K of X, we define the indicator function $I_K \in \Gamma_0(X)$ and the support function $\sigma_K \in \Gamma_0(X^*, w^*)$ as follow:

$$I_K(x) = \begin{cases} 0 & if\ x \in K \\ +\infty & if\ x \notin K. \end{cases} \qquad \sigma_K(x^*) = \sup_{x \in K} \langle x^*, x\rangle.$$

It is direct that $\sigma_K = I_K^*$.

Definition 1.3 (Subdifferential)

For $f \in \Gamma_0(X)$ and $\varepsilon > 0$, we define the multifunction $\partial f, \partial_\varepsilon f : X \rightrightarrows X^*$ given by

$$\partial f(x) = \{x^* \in X^* \ : \ \langle x^*, y - x \rangle + f(x) \leq f(y), \forall y \in X\}$$

$$\partial_\varepsilon f(x) = \{x^* \in X^* \ : \ \langle x^*, y - x \rangle + f(x) \leq f(y) + \varepsilon, \forall y \in X\}$$

Nice observation: $x^* \in \partial_\varepsilon f(x) \iff x \in \partial_\varepsilon f^*(x^*)$, for all $\varepsilon \geq 0$.

Really good books:

[Phe89] R. R. Phelps. *Convex functions, monotone operators, and differentiability (Lecture Notes in Mathematics)*. Springer-Verlag, Berlin New York, 1989.

[Zal02] C. Zalinescu. *Convex analysis in general vector spaces*. World Scientific, River Edge, N.J. London, 2002.

Definition 1.4 (Upper semicontinuity)

Let $M : (T,\tau) \rightrightarrows (Z,\theta)$ a multifunction from a topological space T to a topological vector space Z, and let $t_0 \in T$. We say that

1. *M is τ-θ upper semicontinuous at t_0, if*

$$\forall V \in \theta, [M(t_0) \subseteq V \Longrightarrow \exists U \in \mathcal{N}(t_0),\ M(U) \subseteq V].$$

2. *M is τ-θ Hausdorff-upper semicontinuous at t_0, if*

$$\forall V \in \mathcal{N}(0,\theta), \exists U \in \mathcal{N}(t_0),\ M(U) \subseteq M(t_0) + V.$$

Hausdorff upper-semicontinuity (H-usc) was introduced by Giles, Gregory, and Sims in 1978 for duality maps. Also Contreras and Payá in 1994 provide examples showing that H-usc is strictly weaker that upper semicontinuity. For further references, see

[CP94] M. D. Contreras and R. Payá. *On upper semicontinuity of duality mappings*. Proceedings of the American Mathematical Society, 121(2), 1994.

[CSZ07] A. K. Chakrabarty, P. Shunmunagaraj, and C. Zalinescu. *Continuity properties for the sub-differential and ε-subdifferential of a convex function and its conjugate*. Journal of Convex Analysis, 14(3):479?514, 2007.

We will use the notation of [CSZ07]. In that paper, also the following useful multifunction is introduced:

Definition 1.5 (Multifunction S_{f^*})

Let $f^* \in \Gamma_0(X^*)$, *we define the multifunction*

$$\begin{aligned} S_{f^*} &: \mathbb{R}_+ \times X^* \rightrightarrows X \\ &(\varepsilon, x^*) \mapsto S_{f^*}(0, x^*) = X \cap \partial_\varepsilon f^*(x^*). \end{aligned}$$

Theorem 1.6 (Collier 1976)

X has the RNP $\iff$ for each function $f^* \in \Gamma_0(X^*, w^*)$ with $\mathrm{Int}[\mathrm{dom}\, f^*] \neq \emptyset$, f^* is F-Diff. in a G_δ-dense subset of $\mathrm{Int}[\mathrm{dom}\, f^*]$ (w^*-Asplund).

Observation: When $f^* \in \Gamma_0(X^*, w^*)$ is F-Diff. in x^*, then $Df^*(x^*) \in X$.

We are interested whenever for $x^* \in \mathrm{Int}[\mathrm{dom}\, f^*]$, the equality

$$\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}$$

holds.

Why? Because we have use it as an hypothesis in a previous work:

[CHS14] R. Correa, A. Hantoute, and D. Salas. *Integration of nonconvex epi-pointed functions in locally convex spaces.* (Submitted), 2014.

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Theorem 2.1 (CSZ07)

Let $f \in \Gamma_0(X)$ and $x^* \in \mathrm{Int}[\mathrm{dom}\, f^*]$. The following assertions are equivalent:

(i) *The multifunction $S_{f^*}(\cdot, x^*)$ is τ_0-w H-usc at 0.*

(ii) *The multifunction $S_{f^*}(0, \cdot)$ is $\tau_{\|\cdot\|}$-w H-usc at x^*.*

(iii) *The multifunction S_{f^*} is $(\tau_0 \times \tau_{\|\cdot\|})$-$w$ H-usc at $(0, x^*)$.*

(iv) $\partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}}$.

(v) $(f^*)'(x^*; \cdot)$ *is w^*-lsc.*

Definition 2.2 (SDPD space)

X is an SDPD space if for each function $f \in \Gamma_0(X)$ such that $\mathrm{Int}[\mathrm{dom}\, f^*] \neq \emptyset$, the set

$$WE(f) = \left\{ x^* \in \mathrm{Int}[\mathrm{dom}\, f^*] \ : \ \partial f^*(x^*) = \overline{X \cap \partial f^*(x^*)}^{w^{**}} \right\}$$

is dense in $\mathrm{Int}[\mathrm{dom}\, f^*]$.

Obs: We can't G_δ-density, since Bourgin and Stegall prove that if for each $C \subseteq X$, closed bounded convex set, the support functionals of C is a set of 2nd Baire category, then X has the RNP.

Proposition 2.3

The following assertions are equivalent:

1. X *is an SDPD space.*
2. *For each function* $f \in \Gamma_0(X)$ *with continuous conjugate* f^**,* $WE(f)$ *is dense in* $\mathrm{Int}[\mathrm{dom}\, f^*]$*.*
3. *For each function* $f \in \Gamma_0(X)$ *with real-valued conjugate* f^**,* $WE(f)$ *is dense in* X^**.*

Lemma 2.4

Let $f \in \Gamma_0(X)$ with $\mathrm{Int}[\mathrm{dom}\, f^*] \neq \emptyset$ and $x_0^* \in \mathrm{Int}[\mathrm{dom}\, f^*]$. Let τ be a (Hausdorff) locally convex topology on X in between the weak-topology and the norm-topology. Denote $\sigma = \sigma_{\mathrm{epi}\, f}$. The following assertions are equivalent:

(i) $S_{f^*}(0,\cdot)$ *is* $\tau_{\|\cdot\|}$-τ *H-usc at* x_0^*.

(ii) $S_\sigma(0,\cdot)$ *is* $(\tau_{\|\cdot\|} \times \tau_0)$-$(\tau \times \tau_0)$ *H-usc at* $(x_0^*, -1)$.

(iii) $\forall t > 0$, $S_\sigma(0,\cdot)$ *is* $(\tau_{\|\cdot\|} \times \tau_0)$-$(\tau \times \tau_0)$ *H-usc at* $(tx_0^*, -t)$.

Proposition 2.5

Let $f \in \Gamma_0(X)$ *with* $\mathrm{Int}[\mathrm{dom}\, f^*] \neq \emptyset$. *We have that* $WE(f)$ *is dense in* $\mathrm{Int}[\mathrm{dom}\, f^*]$ *if and only if* $WE(I_{\mathrm{epi}\, f})$ *is dense in* $\mathrm{Int}[\mathrm{dom}\, \sigma_{\mathrm{epi}\, f}]$.

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Definition 3.1 (Faces)

Let $K \subseteq X$ be a nonempty closed convex set. A **face** *F of K is an exposed set of K, namely there exists $x^* \in X^* \setminus \{0\}$ such that for all $x \in F$*

$$\langle x^*, x\rangle = \sigma_K(x^*) > \langle x^*, y\rangle, \ \forall y \in K \setminus F.$$

In such a case we will write $F = F[K, x^*]$.

Definition 3.2 (weakly exposed faces)

Let K be a nonempty convex closed set of X, and $F = F[K, x^*]$ be a face of K. We say that F is **weakly exposed** *by x^* if for all w-neighborhood W of zero, there exists a $\delta > 0$ such that*

$$S(K, x^*, \delta) \subseteq F + W.$$

We will denote by $WE(K)$ the sets of functionals which weakly expose a face of K.

Definition 3.3 (FRNP)

A Banach space X has the Faces-Radon-Nikodým property if for each nonempty closed bounded convex set K of X, $WE(K)$ is dense in X^*.

If $F[K, x^*]$ is a weakly exposed face which is a singleton, then σ_K is G-Diff. at x^* with $D_G\sigma_K(x^*) \in X$. Therefore, if we reduce the same condition of FRNP for weakly exposed faces to weakly exposed points, we get the RNP (this was proven by Banchir and Daniilidis in 2000).

Lemma 3.4 (Key Lemma)

Let $K \subseteq X$ *be a closed convex set such that* $\mathrm{Int}[\mathrm{dom}\, \sigma_K] \neq \emptyset$ *and* $x^* \in \mathrm{Int}[\mathrm{dom}\, \sigma_K]$ *with* $x^* \neq 0$. *Then there exists an equivalent norm* p *on* X, *a point* $x_0 \in X$ *and a neighborhood* U *of* x^* *not containing* 0, *such that*

$$\partial p^*\big|_U = \partial \sigma_K\big|_U + x_0,$$

where p^* *stands for the dual norm on* X^* *associated to* p.

Theorem 3.5

The following assertions are equivalent:

(i) *For each* $K \subseteq X$ *convex and closed such that* $\mathrm{Int}[\mathrm{dom}\,\sigma_K] \neq \emptyset$, $WE(K)$ *is dense in* $\mathrm{Int}[\mathrm{dom}\,\sigma_K]$.

(ii) X *has the FRNP.*

(iii) *For each equivalent norm* p *on* X, $WE(\mathbb{B}_{(X,p)})$ *is dense in* X^*.

Proposition 3.6

X is an SDPD space if and only if $X \times \mathbb{R}$ has the FRNP.

Corollary 3.7

Every closed subspace of a Banach space with the FRNP has the FRNP. Therefore, every closed subspace of an SDPD space is also an SDPD space.

Problem

If X has the FRNP, then is X an SDPD space?

Is the FRNP a 3-space property?

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The statements of Proposition 2.3, Theorem 3.5, Proposition 3.6, Corollary 3.7 and Proposition **??** still hold if we replace in the definitions of SDPD spaces, FRNP and $WE(\cdot)$ the weak-topology for any (Hausdorff) locally convex topology τ with $w \subseteq \tau \subseteq \tau_{\|\cdot\|}$.

The proofs are exactly the same, but using the τ instead of the weak-topology on X for the H-usc of S_{f^*}.

This replacement can be done noting that the equivalences $(i) \iff (ii) \iff (iii)$ int Theorem 2.1 hold for any (Hausdorff) locally convex topology τ in between the weak-topology and the norm-topology.

Let us denote $SE(f)$ the set of functionals in $\mathrm{Int}[\mathrm{dom}\, f^*]$ where $S_{f^*}(0,\cdot)$ is $\tau_{\|.\|}$-$\tau_{\|.\|}$ H-usc.

It is known (see [CSZ07]) that $x^* \in SE(f)$ if and only if f^* is *strongly subdifferentiable* at x^*, namely the limit

$$\lim_{t\to 0^+} \frac{f^*(x^*+th)-f^*(x^*)}{t}$$

exists uniformly in $\mathbb{S}_{X^*}$. This notion was introduced by Contreras and Payá in 1993 and extensively worked in the 90's.

Definition 4.1 (s-SDPD)

A Banach space X is said to be an **s-SDPD** *space if for each function $f \in \Gamma_0(X)$ with $\mathrm{Int}[\mathrm{dom}\, f^*] \neq \emptyset$, $SE(f)$ is dense in* $\mathrm{Int}[\mathrm{dom}\, f^*]$.

Definition 4.2 (s-FRNP)

A Banach space X has the **s-FRNP** *if for each nonempty closed bounded convex set K of X, $SE(K)$ is dense in X^*.*

Example 4.3

1. $SE(\mathbb{B}_{c_0})$ *is dense in* ℓ^1.
2. $WE(\mathbb{B}_{L^1[0,1]})$ *is dense in* $L^\infty[0,1]$.

Proposition 4.4

Consider the following assertions:

(i) X *has the RNP.*

(ii) X *is a s-SDPD space.*

(iii) X *is an SDPD space.*

We have that $(i) \implies (ii) \implies (iii)$.

RNP $\subseteq$ s-FRNP $\subseteq$ FRNP $\subseteq$ Banach spaces.

We will show that **the last inclusion is strict**, which is basically a renorming problem. A really really good book:

[DGZ93] R. Deville, G. Godefroy, and V. Zizler. *Smoothness and renormings in Banach spaces*. Long- man Scientific & Technical Wiley, Harlow, Essex, England New York, NY, 1993.

Proposition 4.5

Let X be a Banach space and p a rotund equivalent norm on X such that

$$\operatorname{ext}\left[\mathbb{B}_{(X,p)}\right] \cap \operatorname{ext}\left[\mathbb{B}_{(X^{**},p^{**})}\right] = \emptyset.$$

Then, X lacks the FRNP.

Corollary 4.6

If X has a copy of c_0, then X lacks the FRNP.

Proof: P. Morris in 1983 provided a renorming of c_0 that meet the condition of Proposition 4.5.

Problem

There exists an space X with the FRNP, which lacks the RNP?