Minimal projections onto hyperplanes in finite dimensional normed spaces

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Minimal projection in Banach spaces

Definition

Let X be a Banach space and Y its closed subspace. Linear mapping $P : X \to Y$ is called a *projection* if $P|_Y = \mathrm{Id}_Y$. By $\mathcal{P}(X, Y)$ we denote the set of all projections from X onto Y.

Definition

The *relative projection constant* of $Y \subset X$ is defined as

$$\lambda(Y, X) = \inf\{||P|| : P \in \mathcal{P}(X, Y)\}.$$

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If a projection $P : X \to Y$ satisfies $||P|| = \lambda(Y, X)$ then P is called a *minimal projection*.

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Minimal projections were extensively studied by many authors in the context of functional analysis and approximation theory. Mainly the problems of existence of minimal projections, uniqueness of minimal projections, finding explicit formulas for minimal projections and estimates of the relative projection constant $\lambda(Y, X)$ were considered. Some of this estimates are concerned with some classical Banach spaces and some of them are of more general nature. One of the most fundamental results of the second type is an old theorem of Bohnenblust on projections onto subspaces of codimension 1 of finite dimensional real normed spaces.

Bohnenblust Theorem

Theorem [Bohnenblust (1938)]

Let X be a real n-dimensional Banach space and let $Y \subset X$ be its $(n-1)$-dimensional subspace. Then $\lambda(Y, X) \leq 2 - \frac{2}{n}$.

This estimation is optimal as the following example shows.

Example

Let $X = \ell_1^n$ or $X = \ell_\infty^n$ and $Y = \ker f$, where $f(x) = x_1 + x_2 + \ldots + x_n$. Then $\lambda(Y, X) = 2 - \frac{2}{n}$.

In the talk we shall be concerned with the case when the relative projection constant is maximal possible or close to maximal.

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Open questions

There are some naturally arising questions related to the theorem of Bohnenblust.

- Is it possible to give a complete characterization of the equality case in the inequality $\lambda(Y, X) \leq 2 - \frac{2}{n}$? Or at least to give some more meaningful equivalent condition?
- Is it possible to determine the maximal possible number of hyperplanes with the maximal projection constant that a given n-dimensional space can have? Or at least to give some upper bound?
- We know that the relative projection constant is bounded by $2 - \frac{2}{n}$ for **every** hyperplane. Is it true that for **some** hyperplane this constant is smaller? In other words, is it possible to determine the value of $\sup_X \inf_{Y \subset X} \lambda(Y, X)$, where X is a real n-dimensional normed space and $Y \subset X$ is a subspace of dimension $(n-1)$?

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Open questions

For the last question, let us remark that if X is a seperable Banach space and $Y \subset X$ is a hyperplane then $\lambda(Y, X) \leq 2$. However, $L_1[0,1]$ is a seperable Banach space such that $\lambda(Y, L_1[0,1]) = 2$ for every hyperplane Y! Thus, in the seperable case it is possible that the relative projection constant is equal to the maximal one for every hyperplane. It is therefore natural to ask if the same situation could occur in the finite dimensional case.

Problem of Bosznay and Garay

This question is actually a variation of an old and unsolved problem posed by Bosznay and Garay.

Question (Bosznay, Garay '86)

Determine the value of $\sup_X \inf_{Y \subset X} \lambda(Y, X)$, where X is a real n-dimensional normed space and $Y \subset X$ is a subspace of dimension at least 2 and at most $n-1$.

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Determine the value of $\sup_X \inf_{Y \subset X} \lambda(Y, X)$, where X is a real n-dimensional normed space and $Y \subset X$ is a subspace of dimension at least 2 and at most $n - 1$.

An equivalent condition for the equality $\lambda(Y, X) = 2 - \frac{2}{n}$.

Theorem

Let X be an n-dimensional normed space and let $Y = \ker f$, where $f \in S_{X^\star}$, be an $(n-1)$-dimensional subspace of X. Then $\lambda(Y, X) = 2 - \frac{2}{n}$ if and only if there exist extreme points $x_1, x_2, \ldots, x_n$ of the unit ball of X such that the following conditions are satisfied

- $f(x_1) = f(x_2) = \ldots = f(x_n)$,
- vectors $x_1, x_2, \ldots, x_n$ are linearly independent,
- if an arbitrary vector $\mathbb{R}^n \ni x = \sum_{i=1}^{n} w_i x_i$ is written in the basis of x_i, then the folowing inequality holds

$$\max_{i=1,2,\ldots,n} \{|w_1 + w_2 + \ldots + w_{i-1} - w_i + w_{i+1} + \ldots + w_n + 1|\} \leq ||x||.$$

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The third condition is equivalent to the fact that for every $1 \leq i \leq n$ points

$$x_1, x_2, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n$$

lie on a facet of the unit ball.

Three-dimensional case

In the three-dimensional setting, if $Y \subset X$ satisfies $\lambda(Y, X) = \frac{4}{3}$ and $x_1,\ x_2,\ x_3$ are like above then

$$\max\{|a+b-c|, |a-b+c|, |-a+b+c|\} \leq ||ax_1+bx_2+cx_3|| \leq |a|+|b|+|c|$$

for all $a, b, c \in \mathbb{R}$.
Note that if a, b, c are not of the same sign then $\max\{|a+b-c|, |a-b+c|, |-a+b+c|\} = |a| + |b| + |c|$ and therefore $||ax_1 + bx_2 + cx_3|| = |a| + |b| + |c|$ in such a case.

Three-dimensional case

We can also give a geometric description of the normed spaces X which posess a hyperplane with the maximal relative projection constant. For simplicity we state it in the three-dimensional case.

Theorem

Let X be a 3-dimensional normed space. The following conditions are equivalent

- There exists a subspace Y of X such that $\dim Y = 2$ and $\lambda(Y, X) = \frac{4}{3}$.
- There exists a non-degenerated linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $C \subset T(B_X) \subset P$,

where B_X is the unit ball of X, C is the octahedron $\{x : |x_1| + |x_2| + |x_3| \leq 1\}$ and P is the parallelotope with set of vertices: $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1), (1, 1, 1), (-1, -1, -1)\}$.

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Maximal projection implies minimal

From the characterization it easily follows that if X has an $(n-1)$-dimensional subspace Y with a maximal possible projection constant then X has also a two-dimensional subspace Z with a minimal possible projection constant. In fact, if $Z = \text{lin}\{x_i, x_j\}$ for $i \neq j$ then $\lambda(X, Z) = 1$. In the three dimensional setting we can obtain a stability version of this phenomen.

Theorem

Let X be a 3-dimensional normed space. Suppose that there exists a subspace Y of X such that $\dim Y = 2$ and $\lambda(Y, X) = \frac{4}{3} - R$ for some $R > 0$. Then there exists a 2-dimensional subspace Z of X such that

$$\lambda(Z, X) \leq 1 + \frac{9(R + \varphi(R))}{4 - 12(R + \varphi(R))},$$

for some explicit function $\varphi(R)$ satisfying the condition $\lim_{R\to 0} \varphi(R) = 0$.

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In fact,

$$\varphi(R) = \frac{\sqrt{\left(\frac{9R}{8-6R}\right)^2 + 4\left(\frac{9R}{8-6R}\right)} - \left(\frac{9R}{8-6R}\right)}{\left(\frac{9R}{8-6R}\right) + 2 - \sqrt{\left(\frac{9R}{8-6R}\right)^2 + 4\left(\frac{9R}{8-6R}\right)}}.$$

As a consequence we obtain some very slight improvement of the trivial bound in the three dimensional case in the problem of Bosznay and Garay.

Corollary

For every three dimensional normed space X there exists its two dimensional subspace $Y \subset X$ such that $\lambda(Y, X) \leq \frac{4}{3} - 0.007$.

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An upper bound on the number of hyperplanes with the maximal projection constant

Yet another consequence of the equivalent condition of the equality $\lambda(Y, X) = 2 - \frac{2}{n}$ is an upper bound on the number of subspaces with the relative projection constant equal to $2 - \frac{2}{n}$.

Theorem

Suppose that the number of the facets ($(n-1)$-dimensional faces) of the unit ball of an n-dimensional normed space X is equal to $N \geq 0$. Then the number of the $(n-1)$-dimensional subspaces Y, for which $\lambda(Y, X) = 2 - \frac{2}{n}$ is at most $\binom{N}{n}$.

In particular, there are only finitely many hyperplanes Y satisfying $\lambda(Y, X) = 2 - \frac{2}{n}$.

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Proof. Let us recall that if Y satisfies $\lambda(Y,X) = 2 - \frac{2}{n}$ then there exist unit vectors $x_1, x_2, \ldots, x_n$ lying in the hyperplane parallel to Y, such that for every $1 \leq i \leq n$ points $x_1, x_2, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n$ lie on a different facet of B_X. Thus, to every such Y there corresponds a set $F(Y)$ of n different facets of X. To prove our theorem it is therefore enough to show that F is an injection. In this purpose, let us suppose the facets in $F(Y)$ are determined by the functionals $f_1, f_2, \ldots f_n \in S_{X^{\star}}$ and $f = \frac{f_1+f_2+\ldots+f_n}{n-2}$. Then for every $1 \leq i \leq n$ we have

$$f(x_i) = \frac{(n-1)-1}{n-2} = 1.$$

This shows that $Y = \ker f$ is uniquely determined by $F(Y)$ and the conclusion follows. □

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Thank you for your attention!