Banach algebras of Calkin type

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(jointly work with P. Motakis and A. Tolias).

Métabief
Banach spaces and optimization: Conference on the occasion of Robert Deville's 60th birthday.
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A résumé

As usual, for a Banach space X, we denote by

$\mathcal{L}(X)$ the space of all bounded linear operators defined on X

$\mathcal{K}(X)$ the spaces of all compact operators defined on X.

Definition

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It is named after J. W. Calkin,

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Two-sided ideals and congruences in the ring of bounded operators in Hilbert space.
Ann. of Math. 42 (1941), no. 2, 839-87.

who proved that the only non-trivial closed ideal of the bounded linear operators on ℓ_2 is the one of the compact operators.

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Question

Given a Banach algebra A, does there exist a Banach space X such that the Calkin algebra of X is isomorphic, as a Banach algebra, to A?

$$A = \mathcal{L}(X)/\mathcal{K}(X)$$

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This was the first time a Calkin algebra of a Banach space could be explicitly described.

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In his PhD thesis, Tabard gives the first infinite dimensional Banach algebra which can be represented as Calkin algebra . Namely,

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for some Banach space $\mathfrak{X}_T$, where $\ell_1(\mathbb{N}_0)$ is endowed with convolution product.

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Let us note that in the Tabard's finite dimensional examples, Banach algebras are not semi-simple.

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the authors where able to prove that all finite dimensional semi-simple complex algebras are of Calkin type. In particular the algebra

$$\mathbb{M}_{n_1}(\mathbb{K}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{K})$$

endowed with point-wise multiplication.

In

P. Motakis - D. Puglisi - D. Zisimopoulou
A hierarchy of Banach spaces with C(K) Calkin algebras.
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it was proved that for every countable compact metric space K there exists a $\mathcal{L}_\infty$-space $\mathfrak{X}$, so that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

$$Cal(\mathfrak{X}) = C(K).$$

In connection with a classical N. Kalton theorem (1974), let us point out that $\mathfrak{X}$ is the first example of Banach space such that

- $\mathcal{K}(\mathfrak{X})$ contains no copy of c_0 and
- $\mathcal{K}(\mathfrak{X})$ is not complemented in $\mathcal{L}(\mathfrak{X})$.

Question (Motakis, Puglisi, Zisimopoulou)

Does there exist a Banach space X such that its Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is reflexive (infinite dimensional)?

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Let us note that reflexivity of $\mathcal{L}(X)$ implies that X must be finite dimensional (J.M. Baker, 1982)

Theorem (Motakis, Puglisi, Tolias, preprint)

There exists a Banach space X *such that* $\mathcal{L}(X)/\mathcal{K}(X)$ *is hereditary indecomposable and quasi-reflexive of order one.*

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Remark

The possibility of such extreme behavior of the quotient $\mathcal{L}(X)/\mathcal{K}(X)$ contrasts the more canonical one of $\mathcal{L}(X)$. The latter space is always decomposable, containing complemented copies of both X and X^*.

Let us recall that Bellenot, Haydon, and Odell (1989) define $J(X)$ the "jamesification" of a Banach space X endowed with a normalized 1-unconditional basis $(x_i)_i$.

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$$\left\|\sum_{i=1}^{\infty} a_i e_i\right\| = \sup\left\{\left\|\sum_{n=1}^{\infty}\left(\sum_{i=k_n}^{m_n} a_i\right) x_{k_n}\right\| : \ 1 \le k_1 \le m_1 < k_2 \le m_2 \ldots\right\}$$

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In particular, $J(\ell_2)$ is the classical James space J.

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$$\mathcal{J}_*(X) = \mathbb{R}s \oplus \overline{span}\{e_i^*\}_i.$$

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It was proved that if X does not contain ℓ_1 then $\mathcal{J}_*(X) = J(X)^*$.As in the classical case, $J(X)$ and $\mathcal{J}_*(X)$ become Banach algebra endowed with the coordinate-wise multiplication with respect to a suitable basis (see A.D. Andrew, W.L. Green *Canad. J. Math*, **32** (1980)).

Theorem (Motakis, Puglisi, Tolias, preprint)

For any space X with a normalized unconditional basis, the spaces $J(X)$ and $\mathcal{J}_*(X)$ are Calkin algebras.

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For any space X with a normalized unconditional basis, the spaces $J(X)$ and $\mathcal{J}_*(X)$ are Calkin algebras.

As consequence, for each $n \in \mathbb{N}$, there is a Banach space Y its Calkin algebra is quasi reflexive of order n.

Theorem (Motakis, Puglisi, Tolias, preprint)

Let X be a Banach space with a normalized unconditional basis endowed with coordinate-wise multiplication.

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Since $c(\mathbb{N})$ and $bv_1(\mathbb{N})$ are isomorphic as Banach spaces to c_0 and ℓ_1 respectively, by a well known theorem of James any non-reflexive Banach space X with unconditional basis is either isomorphic to $X \oplus c_0$ or to $X \oplus \ell_1$, one obtains

Theorem (Motakis, Puglisi, Tolias, preprint)

Every non-reflexive Banach space X with a normalized unconditional basis is isomorphic as Banach space to a Calkin algebra (that contains a complemented ideal isomorphic as Banach algebra to X).

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Theorem (Motakis, Puglisi, Tolias, preprint)

Every reflexive Banach space X with a normalized unconditional basis is isomorphic as Banach algebra to a complemented ideal of a separable quasi-reflexive Calkin algebra.

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Definition

Let X be a Banach space with a Schauder basis $(e_i)_i$ (with associate biorthogonal functionals $(e_i^*)_i$). An operator $T \in \mathcal{L}(X)$ is said to be *diagonal* if for every $i \neq j \in \mathbb{N}$ we have $e_j^*(T(e_i)) = 0$.

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Let us note that $(e_i^* \otimes e_i)_i$ forms a Schauder basis for $\mathcal{K}_{diag}(X)$.

Then the space $\mathbb{R}I \oplus \mathcal{K}_{diag}(X)$ endowed with operator composition is a commutative semi-simple Banach algebra.

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Then the space $\mathbb{R}I \oplus \mathcal{K}_{diag}(X)$ endowed with operator composition is a commutative semi-simple Banach algebra.
It is well known that whenever $(e_i)_i$ is unconditional then $\mathbb{R}I \oplus \mathcal{K}_{diag}(X)$ is isomorphic to $c(\mathbb{N})$.
If the basis is not unconditional then more interesting thing may occur.

In

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Hereditarily indecomposable Banach algebras of diagonal operators.
Israel J. Math. 181 (2011), 65-110.

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Theorem

Suppose X is a Banach space with a Schauder basis $(e_i)_i$. The map $e_i^* \mapsto e_i^* \otimes e_i$ can be extended to an isomorphism between X^* and $\mathcal{L}_{diag}(X)$ if and only if the basis of $(e_i)_i$ dominates the summing basis of c_0 and the norm of X^* is submultiplicative.

Definition

Let X be a Banach space with a Schauder basis $(e_i)_i$. We say that $(e_i)_i$ dominates the summing basis of c_0 with constant C_1, if for every finite sequence of scalars $(\mu_i)_{i=1}^n$,

$$C_1 \cdot \left| \sum_{i=1}^{n} \mu_i \right| \leq \left\| \sum_{i=1}^{n} \mu_i e_i \right\|$$

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holds.

Definition

X^* is said submultiplicative if there is a constant C so that for all scalars $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ we have

$$\left\| \sum_{i=1}^n a_i b_i e_i^* \right\| \leq C \left\| \sum_{i=1}^n a_i e_i^* \right\| \left\| \sum_{i=1}^n b_i e_i^* \right\|.$$

As consequence one obtains

Corollary

Let X be a Banach space with Schauder basis $(e_i)_i$ for which there exist constants C_1, C_2 so that

(i) *for all $n \in \mathbb{N}$, $\| \sum_{i=1}^{n} e_i \| \leq C_1$;*

(ii) *for all scalars $(a_i)_{i=1}^{n}$ and $(b_i)_{i=1}^{n}$*

$$\left\| \sum_{i=1}^{n} a_i b_i e_i \right\| \leq C \left\| \sum_{i=1}^{n} a_i e_i \right\| \left\| \sum_{i=1}^{n} b_i e_i \right\| .$$

Then if $Y = \overline{span}\{e_i^*, i \in \mathbb{N}\}$ the space $\mathbb{R}I \oplus \mathcal{K}_{diag}(Y)$ is isomorphic as Banach algebra to $\mathbb{R}e_\omega \oplus X$ (the unitization of X).

Then one is able to describe all the aforementioned spaces above as $\mathbb{R}I \oplus \mathcal{K}_{diag}(Y)$.

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Now, to close the circle one needs to show the main result of this talk

Theorem (Motakis, Puglisi, Tolias, preprint)

Let X be a Banach space with a Schauder basis. Then there exists a Banach space $\mathcal{Y}_X$ so that $Cal(\mathcal{Y}_X)$ is isomorphic as Banach algebra to $\mathbb{R}I \oplus \mathcal{K}_{diag}(X)$.

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Question

- Does there exist a Banach space X such that its Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is reflexive (infinite dimensional)?

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However, two main questions on this topic remain open

Question

- Does there exist a Banach space X such that its Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is reflexive (infinite dimensional)?
- Does $C(K)$-spaces can be represented as Calkin algebra, for any uncountable compact topological space K?

Best wishes for Robert Devilles 60th birthday ...

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... even if he looks younger than me!

Thanks for your attention.