Variational principles for supinf problems [1]

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[1] Based on joint works with P. Kenderov, D. Gaumont and D. Kamburova

Setting

Problem

We consider the following supinf problem with constraints:

$$(P) \qquad \sup_{x\in X} \inf_{y\in Kx} f(x,y),$$

where X and Y are completely regular topological spaces (e.g. metric spaces), $K : X \rightrightarrows Y$ is a set-valued mapping with nonempty images and $f : X \times Y \to [-\infty, \infty]$ is an extended real-valued function.

Solution

A solution to (P) is every couple $(x_0, y_0) \in X \times Y$ such that $y_0 \in Kx_0$ and

$$f(x_0, y_0) = \inf_{y\in Kx_0} f(x_0, y) = \sup_{x\in X} \inf_{y\in Kx} f(x,y).$$

Examples

"leader-follower" games

f is the utility function of the first (leader) player; $x \in X$ the choice of the first player, then the second player chooses $y \in Kx$, its feasible set. The value of the problem

$$v_f := \sup_{x \in X} \inf_{y \in Kx} f(x, y)$$

expresses the guaranteed utility for the first player.

Stakelberg (two stage optimization)

when for any $x \in X$ we have

$$Kx := \{y' : g(x, y') = \inf_{y \in Y} g(x, y)\},$$

where $g : X \times Y \to \mathbf{R}$ is a given function.

Examples–Unconstrained case

When $Kx = Y$ for all $x \in X$

non-cooperative games

f is the utility of one of the players in a two-person non-cooperative game. Then again the value of the problem

$$v_f := \sup_{x \in X} \inf_{y \in Y} f(x, y)$$

gives the guaranteed utility for the corresponding player.

Minmax problems

When

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) = \inf_{y \in Y} \sup_{x \in X} f(x, y)$$

the problem can be investigated from the point of view of existence of saddle points.

Variational principles

Perturbations with existence of solution

Given $f : X \times Y \to [-\infty, \infty]$ and $K : X \rightrightarrows Y$, find continuous (bounded) functions g in X and h in Y such that $f + g + h$ has a solution for the supinf problem (P).

Perturbations with well-posedness

Find continuous (bounded) functions g in X and h in Y such that the supinf problem (P) for $f + g + h$ is well-posed.

Examples for one variable

Ekeland's principle, Stegall variational principle, Borwein-Preiss smooth principle, Deville-Godefroy-Zizler smooth principle, Choban-Kenderov-R. continuous prinicples. For two variables - results by McLinden.

Well-posedness

Sup-well-posed

The problem (P) is called *sup-well-posed* if the problem to maximize the function $v(\cdot) := \inf_{y\in K(\cdot)} f(\cdot, y)$ is well-posed in the sense of Tykhonov (i.e. every maximizing sequence $(x_n)_n$ $(v(x_n) \to \sup_{x\in X} v(x))$ converges to the unique maximizer of v.

Well-posed

(P) is *well-posed* if every optimizing sequence for (P) converges to some (in fact, unique) solution (x_0, y_0) of (P). A sequence $(x_n, y_n)_n \subset X \times Y$ is called *optimizing* for (P) if:

1. $y_n \in Kx_n$ for every n;
2. $v(x_n) \to v_f = \sup_{x\in X} \inf_{y\in Kx} f(x, y)$;
3. $f(x_n, y_n) \to v_f$

Auxiliary Lemma:1

Z a completely regular topological space (metric space);
$h : Z \to \mathbf{R} \cup \{+\infty\}$; The set $\mathrm{dom}(h) = \{z \in Z : h(z) < \infty\}$ is the *domain* of h. When h takes values in $\mathbf{R} \cup \{-\infty\}$, the set $\mathrm{dom}(h)$ again consists of all points in Z at which h is finite. $h : Z \to \mathbf{R} \cup \{+\infty\}$ is *proper* if its domain is not empty. $C(Z)$ is the Banach space of all continuous, bounded real-valued functions in Z equipped with the uniform norm $||g||_{Z,\infty} := \sup\{|g(z)| : z \in Z\}$, $g \in C(Z)$.

Unconstrained case–a general variational principle

Lemma

(Kenderov-R.) *Let $h : Z \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous function bounded from below. Let $z_0 \in \mathrm{dom}(h)$ and $\varepsilon > 0$ be such that $h(z_0) < \inf_Z h + \varepsilon$. Then, there exists a continuous bounded function $g : Z \to \mathbf{R}_+$, $g(z_0) = 0$, $\|g\|_{Z,\infty} \leq \varepsilon$ and the function $h + g$ attains its minimum in Z at z_0. Moreover, g can be chosen such that $\|g\|_{Z,\infty} = h(z_0) - \inf_Z h$.*

Auxiliary Lemma:2

Constrained case

Lemma

Let $h : Z \to [-\infty, +\infty]$ *be a proper lower semicontinuous function. Suppose that* A *is a closed subset of* Z *such that* $A \cap \operatorname{dom}(h) \neq \emptyset$ *and* h *is bounded from below on* A*. Let* $z_0 \in A \cap \operatorname{dom}(h)$ *and* $\varepsilon > 0$ *be such that* $h(z_0) < \inf_A h + \varepsilon$*. Then, there exists a continuous bounded function* $g : Z \to \mathbf{R}_+$, $g(z_0) = 0$, $\|g\|_{Z,\infty} \leq \varepsilon$ *and the function* $h + g$ *attains its minimum in* A *at* z_0*. Moreover,* g *can be chosen such that* $\|g\|_{Z,\infty} = h(z_0) - \inf_A h$*.*

A well-known result

Lemma

Let $f : X \times Y \to [-\infty, +\infty]$ *be an extended real-valued function such that* f *is upper semicontinuous in* $X \times Y$ *(with the product topology). Let* $K : X \rightrightarrows Y$ *be a lower semicontinuous set-valued mapping with nonempty images. Then the function* $v(\cdot) = \inf_{y \in K(\cdot)} f(\cdot, y)$ *is upper semicontinuous in* X*.*

Assumptions

Consider the following assumptions for the function $f : X \times Y \to [-\infty, \infty]$ and the set-valued mapping $K : X \rightrightarrows Y$:

(A1) f is upper semicontinuous in $X \times Y$;

(A2) the function $v(\cdot) = \inf_{y \in K(\cdot)} f(\cdot, y)$ is bounded above in X and proper as a function with values in $\mathbf{R} \cup \{-\infty\}$;

(A3) for every $x \in X$ the function $f(x, \cdot)$ is lower semicontinuous in Y;

(A4) the set-valued mapping K is lsc (lower semicontinuous) in X with nonempty closed images.

A variational principle for supinf problems with constraints

Theorem

Let $f : X \times Y \to [-\infty, +\infty]$ *be an extended real-valued function and* $K : X \rightrightarrows Y$ *be a set-valued mapping which satisfy (A1)-(A4). Let* $\varepsilon > 0$ *and* $x_0 \in X$ *be such that* $v(x_0) > sup_{x \in X} v(x) - \varepsilon$ *and let* $\delta > 0$ *and* $y_0 \in Kx_0$ *be such that* $f(x_0, y_0) < \inf_{y \in Kx_0} f(x_0, y) + \delta$*. Then, there exist continuous bounded functions* $q : X \to \mathbf{R}_+$ *and* $p : Y \to \mathbf{R}_+$*, such that* $q(x_0) = p(y_0) = 0$*,* $||q||_{X,\infty} \leq \varepsilon$*,* $||p||_{Y,\infty} \leq \delta$ *and the supinf problem* $\sup_{x \in X} \inf_{y \in Kx}\{f(x, y) - q(x) + p(y)\}$ *has a solution at* (x_0, y_0)*.*

If f and the set-valued mapping K satisfy the assumptions (A1)-(A4), and $\varepsilon > 0$ and $\delta > 0$ are arbitrary, then points x_0 and y_0 as in the statement above always exist.

Dense variational principles for supinf problems with constraints

Theorem

Let the function $f : X \times Y \to [-\infty, +\infty]$ *and the mapping* $K : X \rightrightarrows Y$ *satisfy assumptions (A1)-(A4). Then:*

(a) *The set* $\{(q,p) \in C(X) \times C(Y)$: *the function* $f(x,y) + q(x) + p(y)$, $(x,y) \in X \times Y$, *has a solution for the supinf problem*$\}$ *is a dense subset of* $C(X) \times C(Y)$;

(b) *The set* $\{u \in C(X \times Y)$: *the function* $f(x,y) + u(x,y)$, $(x,y) \in X \times Y$, *has a solution for the supinf problem*$\}$ *is a dense subset of* $(C(X \times Y), \| \cdot \|_{X \times Y, \infty})$.

Sup-well-posed perturbations

Proposition

Let $f : X \times Y \to [-\infty, +\infty]$ be an extended real-valued function and $K : X \rightrightarrows Y$ be a set-valued mapping which satisfy (A1)-(A4). Let $\varepsilon > 0$ and $x_0 \in X$ be such that $v(x_0) > sup_{x \in X} v(x) - \varepsilon$ and let $\delta > 0$ and $y_0 \in Kx_0$ be such that $f(x_0, y_0) < \inf_{y \in Kx_0} f(x_0, y) + \delta$. Suppose that x_0 has a countable local base in X (e.g. X is a metric space). Then, there exist continuous bounded functions $q : X \to \mathbf{R}_+$ and $p : Y \to \mathbf{R}_+$, such that $q(x_0) = p(y_0) = 0$, $||q||_{X,\infty} \leq \varepsilon$, $||p||_{Y,\infty} \leq \delta$, the supinf problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) - q(x) + p(y)\}$ has a solution at (x_0, y_0) and the problem is also sup-well-posed with unique sup-solution x_0.

Characterization of well-posedness

Let $S_f : C(X) \times C(Y) \rightrightarrows X \times Y$ be the set-valued mapping which assigns to every functions $q \in C(X)$ and $p \in C(Y)$ the set of solutions (possibly empty) to the problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) + q(x) + p(y)\}$.

Theorem

Let $f : X \times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ satisfy the assumptions (A1)-(A4). Then the mapping S_f is single-valued and upper semicontinuous at $(q, p) \in C(X) \times C(Y)$ if and only if the supinf problem for the function $f(x, y) + q(x) + p(y)$, $(x, y) \in X \times Y$, is well-posed.

Characterization of well-posedness-II

Let $\tilde{S}_f : C(X \times Y) \rightrightarrows X \times Y$ be the set-valued mapping which assigns to every function $u \in C(X \times Y)$ the set of solutions (possibly empty) to the problem $\sup_{x \in X} \inf_{y \in Kx} \{f(x, y) + u(x, y)\}$.

Theorem

Let $f : X \times Y \to [-\infty, +\infty]$ and $K : X \rightrightarrows Y$ satisfy the assumptions (A1)-(A4). Then the mapping $\tilde{S}_f$ is single-valued and upper semicontinuous at $u \in C(X \times Y)$ if and only if the supinf problem for the function $f(x, y) + u(x, y)$, $(x, y) \in X \times Y$, is well-posed.

Variational principle in the unconstrained case

When $Kx = Y$ for any $x \in X$ then a variational principle is valid only with the assumptions

(A2) the function $v(\cdot) = \inf_{y \in Y} f(\cdot, y)$ is bounded above in X and proper as a function with values in $\mathbf{R} \cup \{-\infty\}$;

(A3) for every $x \in X$ the function $f(x, \cdot)$ is lower semicontinuous in Y;

Namely,

Theorem

Let $f : X \times Y \to [-\infty, +\infty]$ be an extended real-valued function which satisfy (A2)-(A3). Let $\varepsilon > 0$ and $x_0 \in X$ be such that $v(x_0) > sup_{x \in X} v(x) - \varepsilon$ and let $\delta > 0$ and $y_0 \in Y$ be such that $f(x_0, y_0) < \inf_{y \in Y} f(x_0, y) + \delta$. Then, there exist continuous bounded functions $q : X \to \mathbf{R}_+$ and $p : Y \to \mathbf{R}_+$, such that $q(x_0) = p(y_0) = 0$, $||q||_{X,\infty} \leq \varepsilon$, $||p||_{Y,\infty} \leq \delta$ and the supinf problem $\sup_{x \in X} \inf_{y \in Y} \{f(x, y) - q(x) + p(y)\}$ has a solution at (x_0, y_0).

Results in the unconstrained case

The following of the above presented results are true in the unconstrained case only with conditions (A2)-(A3)

1. Dense variational principles in $C(X) \times C(Y)$ and in $C(X \times Y)$ with the corresponding norms;
2. A variational principle with sup-well-posedness of the perturbed function;
3. Characterizations of the well-posedness of the problem (P) via the solution mappings S_f and $\tilde{S}_f$.

Thank you!

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Bon Anniversaire Robert!