A sufficient condition for tangential transversality

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Transversality

Transversality of manifolds

Let M_1 and M_2 be two smooth manifolds and $x_0 \in M_1 \cap M_2$. It is said that M_1 and M_2 are transversal at x_0 if the sum of the corresponding tangent spaces at x_0 is the whole space.

A generalization of H. Sussmann (2000)

Let C_1 and C_2 be two convex closed cones. It is said that C_1 and C_2 are transversal (strongly transversal) if $C_1 - C_2$ is the whole space (and $C_1 \cap C_2 \neq \{0\}$).

Separation of sets

The sets A, B containing a point x_0 are said to be locally separated at x_0, if there exists a neighborhood Ω of x_0 so that $\Omega \cap A \cap B = \{x_0\}$.

Transversality and separation of sets

H. Sussmann (2000)

Let $x_0 \in A \cap B$, let C^A and C^B be the approximating cones of the sets A and B of the same type (either Clarke or Boltyanski) which are strongly transversal. If A and B are closed subset of a finite-dimensional space, then they are not locally separated at the point x_0.

Counterexample

Let $A := \{(x_n) \in l_2 : \ |x_n| \leq 1/n\} \subset l_2$ be a Hilbert cube and $B := \{\lambda y : \ \lambda \geq 0\}$ be a ray with $y := (1/n^{3/4})_{n=1}^{\infty}$. Then the corresponding Clarke tangent cones $\hat{T}_A(0) = l_2$ and $\hat{T}_B(0) = B$ are strongly transversal, while the sets A and B are locally separated at the point zero.

Subtransversality

D. Drusvyatsiy, A. D. Ioffe, A. S. Lewis (2015)

Let A and B be closed subsets of the Banach space X. A and B are said to be subtransversal at $x_0 \in A \cap B$, if there exists $K > 0$ such that

$$d(x, A \cap B) \leq K(d(x, A) + d(x, B))$$

for all x in a fixed neighborhood of x_0.

Subtransversality is a key assumption for two types of results: linear convergence of sequences generated by projection algorithms and a qualification condition for normal intersection property with respect to the limiting normal cones and a sum rule for the limiting subdifferentials.

Subtransversality

It is remarkable that subtransversality implies a rather general nonseparation result which is crucial for obtaining necessary optimality conditions of Pontryagin maximum principle type (including optimal control problems with infinite-dimensional state space).

Nonseparation result

Let A and B be closed subsets of the Banach space X. Let A and B be subtransversal at $\mathrm{x}_0 \in \mathrm{A} \cap \mathrm{B}$ with constants $\delta > 0$ and $\mathrm{K} > 0$. Let there exist v^{A} with unit norm which belongs to the Bouligand tangent cone to A at x_0, v^{B} with unit norm which belongs to the derivable tangent cone to B at x_0 and let $\|\mathrm{v}^{\mathrm{A}} - \mathrm{v}^{\mathrm{B}}\| < \frac{1}{\mathrm{K}}$. Then A and B cannot be locally separated at the point x_0.

Moreover, subtransversality is a natural assumption for proving abstract Lagrange multiplier rule.

Lagrange multiplier rule

Let us consider the optimization problem

$$f(x) \to \min \quad \text{subject to} \quad x \in S \ ,$$

where $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper and S is a closed subset of the Banach space X. Let x_0 be a solution of the above problem. Let $\tilde{C}_{\text{epi}f}(x_0, f(x_0))$ and $C_S(x_0)$ be closed convex cones, contained in the corresponding Bouligand approximating cones $T_{\text{epi}f}(x_0, f(x_0))$ and $T_S(x_0)$. Let at least one of them consist of derivable tangent vectors.

Lagrange multiplier rule (continuation)

(a) If $\tilde{\mathrm{C}}_{\mathrm{epif}}(\mathrm{x}_0, \mathrm{f}(\mathrm{x}_0)) - \mathrm{C}_{\mathrm{S}}(\mathrm{x}_0) \times (-\infty, 0]$ is not dense in $\mathrm{X} \times \mathbb{R}$, then there exists a pair $(\xi, \eta) \in \mathrm{X}^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (0, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $\langle \xi, \mathrm{v} \rangle \leq 0$ for every $\mathrm{v} \in \mathrm{C}_{\mathrm{S}}(\mathrm{x}_0)$;

(iv) $\langle \xi, \mathrm{w} \rangle + \eta \mathrm{s} \geq 0$ for every $(\mathrm{w}, \mathrm{s}) \in \tilde{\mathrm{C}}_{\mathrm{epif}}(\mathrm{x}_0, \mathrm{f}(\mathrm{x}_0))$.

(b) If $\tilde{\mathrm{C}}_{\mathrm{epif}}(\mathrm{x}_0, \mathrm{f}(\mathrm{x}_0)) - \mathrm{C}_{\mathrm{S}}(\mathrm{x}_0) \times (-\infty, 0]$ is dense in $\mathrm{X} \times \mathbb{R}$, then epif and $\mathrm{S} \times (-\infty, \mathrm{f}(\mathrm{x}_0)]$ are not subtransversal at $(\mathrm{x}_0, \mathrm{f}(\mathrm{x}_0))$.

Tangential transversality

Our approach is proving tangential transversality instead of subtransversality.

Tangential transversality

Let A and B be closed subsets of the Banach space X. We say that A and B are tangentially transversal at $x_0 \in A \cap B$, if there exist $M > 0$, $\delta > 0$ and $\eta > 0$ such that for any two different points $x^A \in (x_0 + \delta \bar{B}) \cap A$ and $x^B \in (x_0 + \delta \bar{B}) \cap B$, there exists a sequence $\{t_m\}$, $t_m \searrow 0$, such that for every $m \in \mathbb{N}$ there exist $w_m^A \in X$ with $\|w_m^A\| \leq M$ and $x^A + t_m w_m^A \in A$, and $w_m^B \in X$ with $\|w_m^B\| \leq M$, $x^B + t_m w_m^B \in B$, and the following inequality holds true

$$\|x^A - x^B + t_m(w_m^A - w_m^B)\| \leq \|x^A - x^B\| - t_m \eta \,.$$

Open question

Tangential transversality is a stronger condition than subtransversality (Bivas, Krastanov, Ribarska (2018)), but for the time being it is not known if it is strictly stronger or the class of tangentially transversal pairs of sets coincides with the class of subtransversal pairs of sets. It happens that usually tangential transversality is easier to verify than subtransversality when the information known concerns the tangential structure of the sets.

Preliminaries

Throughout the paper if Y is a Banach space, we will denote by B_Y $[\bar{B}_Y]$ its open [closed] unit ball, centered at the origin. The index could be omitted if there is no ambiguity about the space. Let S be a closed subset of Y at $y \in S$.

Bouligand tangent cone $T_S(y)$ to S at y

$$T_S(y) := \left\{ v \in Y : \frac{y_k - y}{\tau_k} \to v \quad \begin{array}{c} \text{for some sequences } y_k \in S, \\ y_k \to y \text{ and } \tau_k > 0, \tau_k \to 0 \end{array} \right\}$$

Preliminaries

Derivable tangent cone $G_S(y)$ to S at y

$$G_S(y) := \left\{ v \in Y : \frac{\xi(\tau_k) - y}{\tau_k} \to v \quad \begin{array}{l} \text{for some vector-valued} \\ \text{function} \xi : [0, \varepsilon] \to S, \\ \xi(0) = y \text{ and for every} \\ \text{choice of a sequence } \tau_k > 0 \\ \text{with } \tau_k \to 0 \end{array} \right\}$$

Clarke tangent cone $\hat{T}_S(y)$ to S at y

$$\hat{T}_S(y) := \left\{ v \in Y : \begin{array}{c} \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \\ \text{such that for every } t \in [0, \delta] \text{ it holds true} \\ \text{that } S \cap (y + \delta \overline{B}) + tv \subset S + t\varepsilon \overline{B} \end{array} \right\}.$$

Uniform tangent set

Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta \bar{B})$ one can find $\lambda > 0$ for which $S \cap (x + t(v + \varepsilon \bar{B}))$ is non empty for each $t \in [0, \lambda]$.

Sequence uniform tangent set

Let S be a closed subset of X and x_0 belong to S. We say that the bounded set $D_S(x_0)$ is a sequence uniform tangent set to S at the point x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta \bar{B})$ one can find a sequence of positive reals $t_m \to 0$ for which $S \cap (x + t_m(v + \varepsilon \bar{B}))$ is non empty for each positive integer m.

Properties

Let S be a closed subset of X and x_0 belong to S. The following are equivalent

1. $D_S(x_0)$ is a uniform tangent set to S at the point x_0
2. $D_S(x_0)$ is a sequence uniform tangent set to S at the point x_0
3. for each $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that for each $v \in D_S(x_0)$ and for each point $x \in S \cap (x_0 + \delta \bar{B})$ the set $S \cap (x + t(v + \varepsilon \bar{B}))$ is non empty for each $t \in [0, \lambda]$.

Properties

Let S be a closed subset of X and let $x_0 \in S$. Let $D_S(x_0)$ be a uniform tangent set to S at the point x_0. Then, the following hold true:

1. the set $c\,D_S(x_0)$ is a uniform tangent set to S at x_0 for each fixed constant $c > 0$;
2. if $D'_S(x_0) \subset D_S(x_0)$, then $D'_S(x_0)$ is a uniform tangent set to S at x_0;
3. if $D'_S(x_0)$ is another uniform tangent set to S at x_0, then $D_S(x_0) \cup D'_S(x_0)$ is a uniform tangent set to S at x_0;
4. the convex closed closure $\overline{\mathrm{co}}\, D_S(x_0)$ of $D_S(x_0)$ is a uniform tangent set to S at x_0
5. if S is convex, then $(S - x_0) \cap M\bar{B}$ is a uniform tangent set to S at x_0 for every $M > 0$.

Compactly epi-Lipschitz (massive) sets

Borwein – Strojwas (1985)

Let A be a closed subset of the Banach space X and $x_0 \in A$. It is said that A is compactly epi-Lipschitz (massive) at x_0, if there exist $\varepsilon > 0$, $\delta > 0$ and a compact set $K \subset X$, such that for all $t \in [0, \delta]$ the following inclusion holds true

$$A \cap (x_0 + \delta \bar{B}) + t\varepsilon \bar{B} \subset A + tK \ .$$

Bivas, Kr., Ribarska (2018)

Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Let A be compactly epi-Lipschitz (massive) and $\hat{T}_A(x_0) - \hat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.

A simpler version

Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$ and:
(i) there exist bounded "ball covering" sets $M_A,\ M_B$ such that $M_A - M_B$ is dense in $\varepsilon \bar{B}$ and "correcting" sets $U_A,\ U_B$ such that

$$A \cap (x_0 + \delta \bar{B}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta \bar{B}) + tM_B \subset B + tU_B$$

whenever $t \in [0, \delta]$;
(ii) there exist uniform tangent sets D_A (to A at x_0) and D_B (to B at x_0) such that $D_A - D_B$ is dense in $U_A - U_B$.
Then A and B are tangentially transversal at x_0.

Main result

Let A and B be closed subsets of the Banach space X and let $x_0 \in A \cap B$. Assume that there exist $\varepsilon > 0$, $\delta > 0$, $q_1 > 0$, $q_2 > 0$, such that $q_1 + q_2 < 1$ and:
(i) there exist bounded "ball covering" sets M_A and M_B such that $M_A - M_B$ is εq_1-dense in $\varepsilon \bar{B}$ and "correcting" sets U_A, U_B such that

$$A \cap (x_0 + \delta \bar{B}) + tM_A \subset A + tU_A \text{ and } B \cap (x_0 + \delta \bar{B}) + tM_B \subset B + tU_B$$

whenever $t \in [0, \delta]$;
(ii) there exist two bounded sets D_A and D_B such that $D_A - D_B$ is εq_2-dense in $U_A - U_B$ and they are "η-uniform" with $\eta := (1 - q_1 - q_2)/3$, i.e. for each $t \in [0, \delta]$

$$A \cap (x_0 + \delta \bar{B}) + tD_A \subset A + t\eta \bar{B} \text{ and } B \cap (x_0 + \delta \bar{B}) + tD_B \subset B + t\eta \bar{B}.$$

Then A and B are tangentially transversal at x_0.

Borwein – Strojwas condition

Definition

Let X and Y be Banach spaces and $f : X \times Y \to R \cup \{+\infty\}$ be a proper lower semicontinuous function which has finite value at $(\bar{x}, \bar{y}) \in X \times Y$. It is said that f satisfies the Borwein – Strojwas condition at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$ iff there exist positive reals $\bar{\delta} > 0$ and $K > 0$ such that for every $t \in [0, \bar{\delta}]$ the following inclusion holds true:

$$\operatorname{epi} f \cap \left((\bar{x}, \bar{y}, f(\bar{x}, \bar{y})) + \bar{\delta} \cdot \bar{B}_{X\times Y\times R}\right) + t\left(\bar{B}_X, 0, 0\right) \subset$$

$$\subset \operatorname{epi} f + t\left(0, K \cdot \bar{B}_Y, K[-1, 1]\right).$$

Remark

This condition is related to the pseudo-Lipshits condition used by Clarke (2004) for studying of the basic problem of the calculus of variations.

The main motivation of our research

Theorem

Let X and Y be Banach spaces and $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function which satisfies the Borwein – Strojwas condition at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$. Let $L : Y \longrightarrow X$ be a compact linear operator and $S := \{(Ly, y) : y \in Y\}$. We assume that

$$\hat{T}_{\operatorname{epi} f}(\bar{x}, \bar{y}, f(\bar{x}, \bar{y})) - S \times (-\infty, 0]$$

is dense in $X \times Y \times \mathbb{R}$. Then $\operatorname{epi} f$ and $S \times (-\infty, f(\bar{x}, \bar{y})]$ are tangentially transversal at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$.

Corollary 1.

Let X and Y be Banach spaces. We consider the optimization problem

$$f(x,y) \to \min \quad \text{subject to} \quad (x,y) \in S ,$$

where $f : X \times Y \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, proper and satisfies the Borwein condition at $(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$ and $S := \{(Ly, y) : y \in Y\}$, where $L : Y \longrightarrow X$ is a compact linear operator. Let $(\bar{x}, \bar{y})$ be a solution of the above problem. Then there exists a triple $(\xi, \eta, \zeta) \in X^* \times Y^* \times \mathbb{R}$ such that

(i) $(\xi, \eta, \zeta) \neq (0, 0, 0)$;

(ii) $\zeta \in \{0, 1\}$;

(iii) $\langle \xi, Ly \rangle + \langle \eta, y \rangle = 0$ for every $y \in Y$;

(iv) $\langle \xi, u \rangle + \langle \eta, v \rangle + \zeta w \geq 0$ for every $(u, v, w) \in \hat{T}_{\text{epif}}(\bar{x}, \bar{y}, f(\bar{x}, \bar{y}))$.

Corollaries

Definition

Let A and B be closed subsets of the Banach space X and $x_0 \in A \cap B$. We say that A and B are jointly massive at x_0 if there exist $\varepsilon > 0$, $\bar{\delta} > 0$, bounded sets $M_A \subset X$, $M_B \subset X$ and a compact set $K \subset X$ such that:
(i) $\varepsilon \bar{B}_X \subset \overline{M_A - M_B}$;
(ii) $A \cap (x_0 + \bar{\delta}\bar{B}) + tM_A \subset A + tK$ and $B \cap (x_0 + \bar{\delta}\bar{B}) + tM_B \subset B + tK$ whenever $t \in [0, \bar{\delta}]$.

Corollary 2.

Let A and B be jointly massive at x_0 and $\hat{T}_A(x_0) - \hat{T}_B(x_0)$ be dense in X. Then A and B are tangentially transversal at x_0.

Corollary 3.

Let us consider the optimization problem

$$\mathrm{f}(\mathrm{x}) \to \min \quad \text{subject to} \quad \mathrm{x} \in \mathrm{S} \;,$$

where $\mathrm{f} : \mathrm{X} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous and proper, and S is a closed subset of the Banach space X. Let epif and $\mathrm{S} \times (-\infty, \mathrm{f}(\mathrm{x}_0)]$ be jointly massive at the point $(\mathrm{x}_0, \mathrm{f}(\mathrm{x}_0))$. Then there exists a pair $(\xi, \eta) \in \mathrm{X}^* \times \mathbb{R}$ such that

(i) $(\xi, \eta) \neq (0, 0)$;

(ii) $\eta \in \{0, 1\}$;

(iii) $\langle \xi, \mathrm{v} \rangle \; \leq 0$ for every $\mathrm{v} \in \hat{\mathrm{T}}_{\mathrm{S}}(\mathrm{x}_0)$;

(iv) $\langle \xi, \mathrm{w} \rangle + \eta \mathrm{s} \geq 0$ for every $(\mathrm{w}, \mathrm{s}) \in \hat{\mathrm{T}}_{\mathrm{epif}}(\mathrm{x}_0, \mathrm{f}(\mathrm{x}_0))$.

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