Preservance of octahedrality and strong diameter two properties by tensor product spaces

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(joint with J. Langemets and V. Lima)

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holds for every $y \in Y$ and $\lambda \in \mathbb{R}$.

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Theorem (Godefroy, 1989)

Let X be a Banach space. Then X admits an equivalent octahedral norm if, and only if, X contains an isomorphic copy of ℓ_1.

Preliminary definitions II: Tensor product spaces

Given two Banach spaces X and Y we will denote by $X\widehat{\otimes}_\varepsilon Y$ the injective tensor product of X and Y, which is nothing but the completion of the weak*-to-weak continuous finite-rank operators from Y^* to X (which we will denote by $F(Y^*, X)$).

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It is well known that $(X\widehat{\otimes}_\varepsilon Y)^* = X^*\widehat{\otimes}_\pi Y^*$ when X^* or Y^* has the approximation property and X^* or Y^* has the RNP.

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1. It is known that X has an octahedral norm if, and only if, every convex combination of weak-star slices of B_{X^*} has diameter two (X^* has the w^*-SD2P) (J. Becerra Guerrero, G. López-Pérez and A.R.Z. (2014)).

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3. If X and Y have the SD2P, so does $X\widehat{\otimes}_\pi Y$ (J. Becerra Guerrero, G. López-Pérez and A.R.Z. (2015)).

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If one consider **in a very informal way** that $(X\widehat{\otimes}_\varepsilon Y)^* = X^*\widehat{\otimes}_\pi Y^*$, one should expect that octahedrality is preserved by injective tensor product.

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Which is in turn part of a more general problem.

Problem (T. Abrahamsen, V. Lima and O. Nygaard (2013))

How is SD2P preserved by tensor product spaces?

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Let X and Y be Banach space and assume that Y^* is uniformly convex.

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Theorem (J. Langemets, V. Lima and A. R. Z. (to appear))

Let X and Y be Banach space and assume that Y^* is uniformly convex. If $X\widehat{\otimes}_\varepsilon Y$ has an octahedral norm then Y^* is finitely representable in X.

Sketch of the proof

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$$\|y_i\otimes x+T\|>2-\delta(\varepsilon)\ \forall i.$$

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$$2 - \delta(\varepsilon) < \|\varphi_i(y_i)x + T(\varphi_i)\| \leq |\varphi_i(y_i)| + \|T(\varphi_i)\|.$$

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Using the uniform convexity assumption we deduce that $\|f_i - \varphi_i\| < \varepsilon$, and thus $\|T(f_i)\| > 1 - \delta(\varepsilon) - \varepsilon$. Since $\{f_i\}$ is a γ-net we get $\|T(f)\| > 1 - \delta(\varepsilon) - \varepsilon - \gamma$ in the whole sphere, so $T_{|F}$ is a $(1 + \delta(\varepsilon) + \varepsilon + \gamma)$-isometry.

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Corollary

Let X be ℓ_1 or L_1 and Y be ℓ_p^3 for $1 < p < 2$. Then:

1. $X \widehat{\otimes}_\varepsilon Y$ *does not have an octahedral norm though X does have an octahedral norm.*
2. $X^* \widehat{\otimes}_\pi Y^*$ *does not have the SD2P.*

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Examples of Banach spaces with the Daugavet property are $C([0,1])$, $L_1([0,1])$, $L_\infty([0,1])$, $Lip(M)$ where M denotes a length space.
It is known that if a Banach space X has the Daugavet property then both X and X^* have an octahedral norm (V. Kadets, R. Shvidkoy, G. Sirotkin and D. Werner, 2001).

The Daugavet property II

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If X and Y have the Daugavet property, does $X\widehat{\otimes}_\varepsilon Y$ or $X\widehat{\otimes}_\pi Y$ have the Daugavet property?

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New ideas are welcome!

Thank you