Faculty of Electrical Engineering
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On densely isomorphic normed spaces

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(Joint project with S. Dantas and P. Hájek)

Banach spaces and optimization
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MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY

The starting question

Problem

How different can two dense subspaces of a Banach space be?

In this talk, subspaces are **NOT** assumed to be closed.

Example 1: c_{00} and c_0 (as dense subspaces of c_0).

- c_{00} is meager, has a countable Hamel basis, is 'very small';
- c_0 is a Baire space.

Example 2: An incomplete normed space X and its completion $\widetilde{X}$.

Example 3: c_{00} and $(\ell_1, \|\cdot\|_\infty)$ (as dense subspaces of c_0).

- There is a (non-equivalent) complete norm on $(\ell_1, \|\cdot\|_\infty)$;
- There is no complete norm on c_{00}.

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Setting the scene

Definition

Two normed spaces X and Y are *densely isomorphic* if there exist dense subspaces X_0 of X and Y_0 of Y such that X_0 and Y_0 are isomorphic.

- Isomorphic normed spaces are densely isomorphic, *a fortiori*;
- Every normed space X is densely isomorphic to its completion $\widetilde{X}$;
- If X and Y are densely isomorphic, then $\widetilde{X}$ and $\widetilde{Y}$ are isomorphic;
- Two densely isomorphic <u>Banach</u> spaces are isomorphic.

In particular, given densely isomorphic normed spaces, we can assume that they are dense subspaces of the same Banach space.

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Let Y and Z be dense subspaces of a Banach space X. Must Y and Z be densely isomorphic?

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Let Y and Z be dense subspaces of a Banach space X. Must Y and Z be densely isomorphic?

The separable case

Theorem A

Let Y and Z be dense subspaces of a separable Banach space X. Then Y and Z are densely isomorphic.

Proof. We apply a perturbation argument to an M-basis of X.

- Let $\{e_j; e_j^*\}_{j=1}^\infty$ be a bounded M-basis for X and $(\varepsilon_j)_{j=1}^\infty \searrow 0$;
- Find $(y_j)_{j=1}^\infty \subseteq Y$ and $(z_j)_{j=1}^\infty \subseteq Z$ with $\|y_j - e_j\|, \|z_j - e_j\| < \varepsilon_j$;
- As in the proof of the small perturbation lemma, we prove that $Y_0 := \mathrm{span}(y_j)_{j=1}^\infty$ and $Z_0 := \mathrm{span}(z_j)_{j=1}^\infty$ are isomorphic;
- Finally, Y_0 and Z_0 are dense in X.

Remark

Any two dense subspaces Y and Z of $\ell_1(\Gamma)$ are densely isomorphic.

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Let Y and Z be dense subspaces of a <u>separable</u> Banach space X. Then Y and Z are densely isomorphic.

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Any two dense subspaces Y and Z of $\ell_1(\Gamma)$ are densely isomorphic.

The non-separable nature of the problem

A Banach space X is *weakly Lindelöf determined* (hereinafter, *WLD*) if the dual ball B_{X^*} is a Corson compact in the relative w^*-topology.

For our purposes, X is WLD if it admits an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ that *countably supports* X^*, *i.e.*,

$$\operatorname{supp} x^* := \{\gamma \in \Gamma \colon \langle x^*, x_\gamma \rangle \neq 0\}$$

is a countable subset of Γ, for every $x^* \in X^*$.

Theorem B

Let X be a WLD Banach space such that $\omega_1 \leqslant \operatorname{dens} X \leqslant \mathfrak{c}$. Then there exists a closed subspace X_0 of X, with $\operatorname{dens} X_0 = \operatorname{dens} X$, that contains two dense subspaces Y and Z which are not densely isomorphic.

Corollary. There exist two dense subspaces Y and Z of the Hilbert space $\ell_2(\omega_1)$ that are not densely isomorphic.

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Finding the subspaces

- Let $\{e_\gamma; e^*_\gamma\}_{\gamma<\Gamma}$ be a normalised M-basis for X ($\Gamma := \operatorname{dens} X$);
- Pick an injective long sequence $(q_\gamma)_{\omega\leqslant\gamma<\Gamma}$ in $(0,1)$;
- Set

$$\tilde{e}_\gamma := e_\gamma + \sum_{j=1}^{\infty} (q_\gamma)^j \cdot e_j \qquad (\omega \leqslant \gamma < \Gamma);$$

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- **Fact 1.** $(e^*_n)_{n=1}^{\infty}$ separates points on Y (Vandermonde matrices);
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- **Fact 2.** No dense subspace of Z admits a separating sequence of functionals.

Therefore, Y and Z are not densely isomorphic.

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Therefore, Y and Z are not densely isomorphic.

Finding the subspaces

- Let $\{e_\gamma; e^*_\gamma\}_{\gamma<\Gamma}$ be a normalised M-basis for X ($\Gamma := \operatorname{dens} X$);
- Pick an injective long sequence $(q_\gamma)_{\omega\leqslant\gamma<\Gamma}$ in $(0,1)$;
- Set
$$\tilde{e}_\gamma := e_\gamma + \sum_{j=1}^{\infty} (q_\gamma)^j \cdot e_j \qquad (\omega \leqslant \gamma < \Gamma);$$
- $X_0 := \overline{\operatorname{span}}\{\tilde{e}_\gamma\}_{\omega\leqslant\gamma<\Gamma}$ (a WLD Banach space);
- $Y := \operatorname{span}\{\tilde{e}_\gamma\}_{\omega\leqslant\gamma<\Gamma}$;
- **Fact 1.** $(e^*_n)_{n=1}^{\infty}$ separates points on Y (Vandermonde matrices);
- $Z := \operatorname{span}\{v_\alpha\}_{\alpha<\Gamma}$, where $\{v_\alpha; \varphi_\alpha\}_{\alpha<\Gamma}$ is an M-basis for X_0;
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A stronger result, under CH

Theorem C

(CH) *Let X be a WLD Banach space with $\operatorname{dens} X = \omega_1$. Then there exists a dense subspace Y of X that contains no uncountable biorthogonal system.*

Particular case. (CH) There exists a dense subspace of the Hilbert space $\ell_2(\omega_1)$ that contains no uncountable biorthogonal system.

Lemma

Let $\{e_\alpha; e_\alpha^*\}_{\alpha\in\Gamma}$ be an M-basis for a Banach space X. Then every non-separable subspace of $Z := \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ contains an uncountable biorthogonal system.

Therefore, no non-separable subspace of Y is isomorphic to a subspace of Z (and, in particular, Y and Z are not densely isomorphic).

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Thank you for your attention!