Kalton meets Fourier

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University of Extremadura

Conference on Banach spaces and optimization
on the ocassion of Robert Deville's 60th birthday

Métabief, 16-21th June, 2019

This work was supported by project IB16056, FEDER funds and Junta de Extremadura (Spain).

1. Twisted sums of Banach spaces
2. Symmetries in twisted sums

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☞ *Question:* are there any non-trivial twisted sums of Y and X?

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4. Complemented subspaces of ℓ_1:

$$0 \longrightarrow \ker q \longrightarrow \ell_1 \stackrel{q}{\longrightarrow} \ell_2 \longrightarrow 0$$

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$$Z = Y \oplus X \quad , \quad \|(x,y)\|_\Omega = \|y - \Omega x\|_Y + \|x\|_X$$

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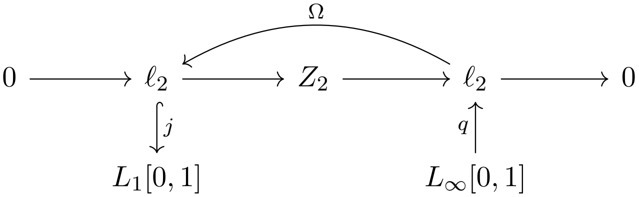

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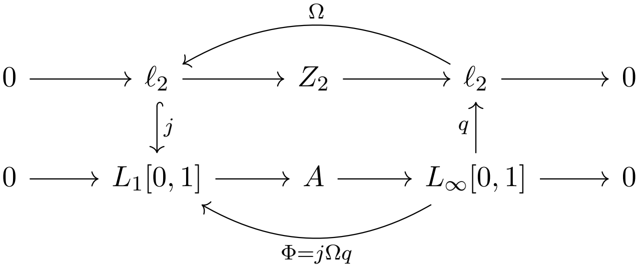

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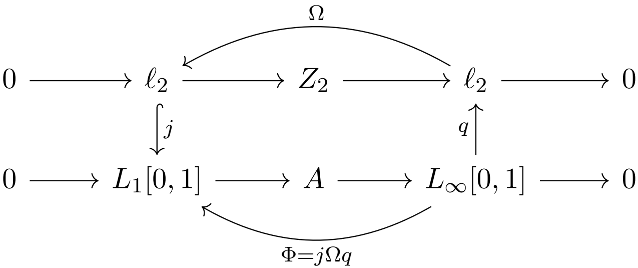

Theorem 1 (Cabello, Castillo, Kalton and Yost)

Φ *is non-trivial.*

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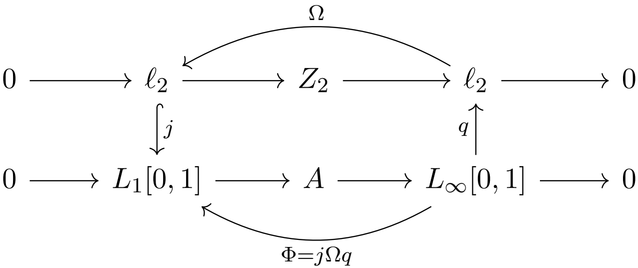

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Theorem 2 (Cabello, Castillo, Kalton and Yost)

There is a non-trivial twisted sum of ℓ_1 *and* c_0*.*

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  We know there are non-trivial, but that is all.

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(1) *Automorphisms of* $\mathbb{N}$*:*

$$\|\Omega(\sigma x) - \sigma(\Omega x)\| \leq M\|x\|$$

where $\sigma x(n) = x(\sigma(n))$.

(2) *Pointwise product:*

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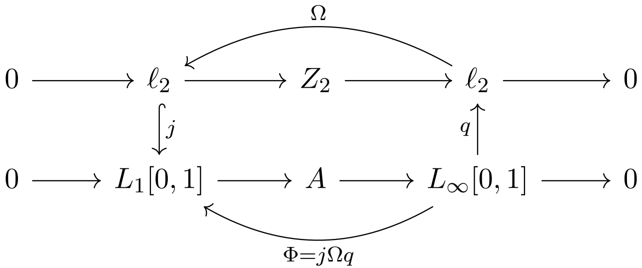
- Ω does not increase supports:

$$\operatorname{supp}(\Omega f) \subset \operatorname{supp} f$$

Pursuing symmetries



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 & & \big\downarrow j & & & & \big\uparrow q & & \\
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$$\Omega : \ell_2 \to \ell_2, \qquad \Phi = j\Omega q : L_\infty[0,1] \to L_1[0,1]$$

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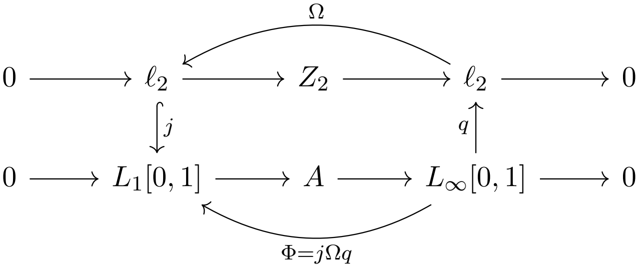

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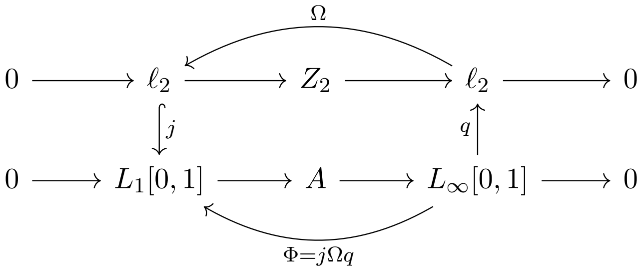

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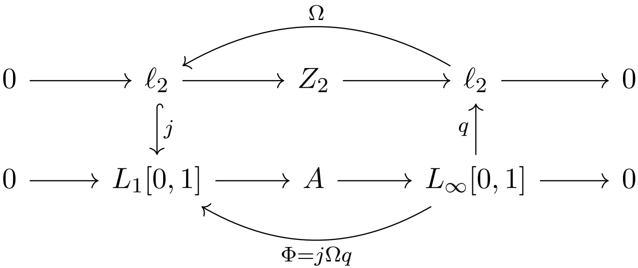

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Theorem 3 (Cabello)

If $\Phi : L_\infty \to L_1$ *respects the pointwise product, then it is trivial.*

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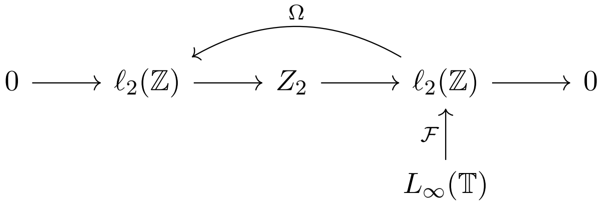

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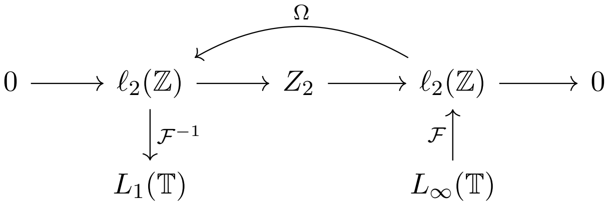

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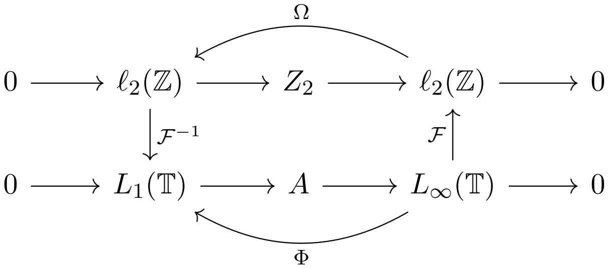

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Theorem 4

If Ω is the Kalton-Peck map, then Φ is non trivial.

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The twisted sum induced by Φ *inherits convolution.*

THANK YOU
FOR YOUR ATTENTION