Coarse Quotient Mappings between Metric Spaces

Sheng Zhang

Department of Mathematics

Texas A&M University

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Linear Quotient Mapping

Let f be a bounded linear operator from a Banach space X onto a Banach space Y. Then by the well-known Open Mapping Theorem f must be open, i.e., $\exists \delta > 0$ such that

$$f(B_X) \supset \delta B_Y.$$

Such a (linear) mapping is a quotient mapping in the topological sense, hence is called a *linear quotient mapping*.

Uniform and Lipschitz Quotient Mapping

Definition

A mapping $f : X \to Y$ between two metric spaces X and Y is called *co-uniformly continuous* if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ such that $\forall x \in X$,

$$f(B(x, \epsilon)) \supset B(f(x), \delta).$$

If δ can be chosen as ϵ / C for some $C > 0$ independent of ϵ, then f is said to be co-Lipschitz.

A mapping $f : X \to Y$ that is both uniformly continuous and co-uniformly continuous (resp. Lipschitz and co-Lipschitz) is called a *uniform (resp. Lipschitz) quotient mapping*. If in addition f is surjective, then Y is called a *uniform (resp. Lipschitz) quotient* of X.

Local Version of a Dual Argument

By using a tool called the uniform approximation by affine property (UAAP), Bates, Johnson, Lindenstrauss, Preiss and Schechtman proved the following theorem:

Theorem (BJLPS, 1999)

X and Y are two Banach spaces. Assume that X is super-reflexive and Y is a uniform quotient of X. Then Y^* is crudely finitely representable in X^*.

Corollary

(i) *A Banach space that is a uniform quotient of a Hilbert space must be isomorphic to a Hilbert space.*

(ii) *If a Banach space Y is a uniform quotient of L_p, $1 < p < \infty$, then Y is isomorphic to a linear quotient of L_p.*

Lipschitz and Uniform Quotients of ℓ_p

By studying the existence of point of ε-Fréchet differentiability of a Lipschitz mapping between Banach spaces, Johnson, Lindenstrauss, Preiss and Schechtman showed that

Theorem (JLPS, 2001)

ℓ_p cannot be a Lipschitz quotient of ℓ_q for $1 < p < q < \infty$.

Later, V. Lima and N. L. Randrianarivony applied the geometric property (β) introduced by S. Rolewicz to prove the following:

Theorem (LR, 2012)

ℓ_q cannot be a uniform quotient of ℓ_p for $1 < p < q < \infty$.

Some Definitions from Coarse Geometry

Definition

A mapping $f : X \to Y$ between two metric spaces X and Y is called *coarsely continuous* if $\forall R > 0$, $\exists S = S(R) > 0$ such that

$$d(x, y) < R \Rightarrow d(f(x), f(y)) < S.$$

Definition

A coarsely continuous mapping $f : X \to Y$ between two metric spaces X and Y is said to be a *coarse equivalence* if there exists another coarsely continuous mapping $g : Y \to X$ such that

$$\sup\{d(g \circ f(x), x) : x \in X\} < \infty,$$
$$\sup\{d(f \circ g(y), y) : y \in Y\} < \infty.$$

In this case we say that X is *coarsely equivalent* to Y. If X is coarsely equivalent to a subset of Y we say that X *coarsely embeds* into Y.

Coarse Quotient Mapping

Definition (Z)

Let $K \geq 0$ be a constant. A mapping $f : X \to Y$ between two metric spaces X and Y is called *co-coarsely continuous* with constant K if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ so that $\forall x \in X$,

$$B(f(x), \epsilon) \subset f(B(x, \delta))^K.$$

If in addition f is coarsely continuous, then f is called a *coarse quotient mapping* with constant K and Y is said to be a *coarse quotient* of X.

Notation: for a subset A of a metric space X we denote by $A^K := \{x \in X : d(x, a) \leq K \text{ for some } a \in A\}$ its K-neighborhood.

Johnson's Lemma

In general, coarse quotient mappings are not necessarily surjective. It only implies that $Y = f(X)^K$. However the next lemma shows that in the Banach space setting one can always assume that $K = 0$.

Lemma (Johnson)

Let X and Y be Banach spaces. Assume that Y is a coarse quotient of X. Then there exists a coarse quotient mapping with constant 0 from X onto Y.

"Co-Lipschitz for Large Distances"

Lemma (Z)

Let X and Y be two metric spaces and $f : X \to Y$ be a co-coarsely continuous mapping with constant K. Assume that Y is metrically convex. Then $\forall \epsilon > 2K$, $\exists C = C(\epsilon) > 0$ so that $\forall x \in X$ and $\forall r \geq \epsilon$,

$$B(f(x), r) \subset f(B(x, \frac{r}{C}))^K.$$

Recall: A metric space X is called metrically convex if $\forall x_0, x_1 \in X$ and $\forall 0 < t < 1$, $\exists x_t \in X$ such that

$$d(x_0, x_t) = td(x_0, x_1) \text{ and } d(x_1, x_t) = (1 - t)d(x_0, x_1).$$

An Ultraproduct Technique

Theorem (Z)

Let X and Y be Banach spaces and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. If Y is a coarse quotient of X, then $Y_{\mathcal{U}}$ is a Lipschitz quotient of $X_{\mathcal{U}}$.

Proof.

Let $f : X \to Y$ be a coarse quotient mapping with constant $K = 0$. Then $\exists L, C > 0$ such that $\forall x \in X$ and $\forall r \geq 1$,

$$f(B(x,r)) \subset B(f(x), Lr) \quad \text{and} \quad B(f(x), r) \subset f(B(x, \frac{r}{C})).$$

For each $n \in \mathbb{N}$ define $f_n : X \to Y$ by $f_n(x) = f(nx)/n$.

Then $T : X_{\mathcal{U}} \to Y_{\mathcal{U}}$ defined by $T((x_n)_{\mathcal{U}}) = (f_n(x_n))_{\mathcal{U}}$ is a Lipschitz quotient mapping from $X_{\mathcal{U}}$ onto $Y_{\mathcal{U}}$. □

Local Version of a Dual Argument

Theorem (Z)

X and Y are two Banach spaces. Assume that X is super-reflexive and Y is a coarse quotient of X. Then Y^* is crudely finitely representable in X^*.

Corollary

(i) *A Banach space that is a coarse quotient of a Hilbert space must be isomorphic to a Hilbert space.*

(ii) *If a Banach space Y is a coarse quotient of L_p, $1 < p < \infty$, then Y is isomorphic to a linear quotient of L_p.*

Property (β)

Definition (Rolewicz, 1987)

A Banach space X is said to have *property (β)* if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ so that $\forall x \in B_X$ and $\forall \{x_n\}_{n=1}^{\infty} \subset B_X$ with $\mathrm{sep}(\{x_n\}) := \inf_{n \neq m} \|x_n - x_m\| \geq \epsilon$, $\exists i$ such that

$$\left\|\frac{x + x_i}{2}\right\| \leq 1 - \delta.$$

The (β)-modulus of X, denoted by $\bar{\beta}_X(\epsilon)$, is the supremum of all $\delta > 0$ so that the above property is satisfied.

The (β)-modulus is said to have power type p if $\exists A > 0$ such that $\bar{\beta}_X(\epsilon) \geq A\epsilon^p$ for all ϵ.

Remark: $\bar{\beta}_{\ell_p}$ has power type p for $1 < p < \infty$.

Coarse Quotients of ℓ_p $(1 < p < \infty)$

Theorem (Z)

Let X and Y be Banach spaces and $1 < p < q < \infty$. Assume that the (β)-modulus of X has power type p and Y contains a subspace isomorphic to ℓ_q. Then there is no coarse quotient mapping from any subset of X to Y that is Lipschitz for large distances.

Corollary

ℓ_q cannot be a coarse quotient of ℓ_p for $1 < p < q < \infty$.

Lemma (Z)

Let X, Y and Z be metric spaces. Assume that Y is a coarse quotient of X and there is no coarse quotient mapping from any subset of X to Z. Then Z does not coarsely embed into Y.

Coarse Quotient of ℓ_p $(1 < p < 2)$

Theorem (Johnson & Odell, 1974)

If a Banach space Y is a subspace of L_p $(2 < p < \infty)$ such that no subspace of Y is isomorphic to ℓ_2, then Y is isomorphic to a subspace of ℓ_p.

Corollary

If a Banach space Y is a coarse quotient of ℓ_p, $1 < p < 2$, then Y is isomorphic to a linear quotient of ℓ_p.

Open Problem

Is ℓ_p a uniform (coarse) quotient of ℓ_q for $2 \leq p < q < \infty$?

Thank You!