A topological characterization of dual strict convexity in Asplund spaces

Banach spaces and optimization: Conference on the occasion of Robert Deville's 60th birthday

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The $(*)$ property and $(*)$ with slices

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Likewise for subspaces (B, w^*) of X^*.

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In general, the slice condition in (2) and (3) is necessary: there exists X such that (X, w) has $(*)$, yet X admits no strictly convex norm (OST 2012).

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1. X^* admits a strictly convex dual norm;
2. X^* admits a dual norm $|||\cdot|||$ such that $(S_{(X^*,|||\cdot|||)}, w^*)$ has a G_δ-diagonal.

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The implication (4) $\Rightarrow$ (1) is the new part.

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Given $x \in S_X$, $a \in \mathbb{R}$, define the w^*-open halfspace

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We index a derivation process by the tree of finite sequences

$$(\mathbb{N} \times \mathbb{Q})^{<\mathbb{N}} := \{s = ((j_0, q_0), \ldots, (j_{n-1}, q_{n-1})) : j_k \in \mathbb{N}, q_k \in \mathbb{Q} \text{ and } n \in \mathbb{N}\}.$$

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$$B_{s^\frown(j,q)} = \bigcap \{B_s \setminus H_{x,a+q} : x \in S_X,\ a > -2 \text{ and } B_s \cap H_{x,a} \subseteq U \text{ some } U \in \mathscr{U}_j\}.$$

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$$B_{X^*} = B_{s_0} \supseteq B_{s_1} \supseteq B_{s_2} \supseteq \ldots \quad \text{and} \quad f, g \in B := \bigcap_{n=0}^{\infty} B_{s_n}.$$

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We apply Choquet's Lemma to d and e to show that $(\mathscr{V}_j)$ is not a $(*)$-sequence for (S_{X^*}, w^*): given $j \in \mathbb{N}$, either

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2. $d, e \in V$ for some $V \in \mathscr{V}_j$.