# Decompositions of variable Lebesgue norms by ODE techniques

## Septièmes journées "Besançon-Neuchâtel" d'Analyse Fonctionnelle

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# Abstract

We study decompositions of Nakano type varying exponent Lebesgue norms and spaces. These function spaces are represented here in a natural way as tractable varying $\ell^p$ sums of projection bands. The main results involve embedding the varying Lebesgue spaces to such sums, as well as the corresponding isomorphism constants.

The main tool applied here is an equivalent variable Lebesgue norm which is defined by a suitable ordinary differential equation introduced recently by the author.

There is a preprint in the ArXiv with a similar title.

# Nakano spaces

- Variable Lebesgue spaces, or varying exponent $L^p$ spaces are often equipped with the following Musielak-Orlicz type norm:

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where

$$\rho(g) = \int_0^1 |g(t)|^{p(t)}\, dt, \quad p, f, g \in L^0[0,1],\ p \geq 1.$$

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- In the paper we investigate Nakano $L^{p(\cdot)}$ norms:

$$|||f|||_{p(\cdot)} = \inf\left\{\lambda > 0\colon \int_0^1 \frac{1}{p(t)} \left|\frac{f(t)}{\lambda}\right|^{p(t)}\ dt \leq 1\right\}$$

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(here the terminology varies in the literature).
- The above norms are derived from the Minkowski functional and are special cases of Musielak-Orlicz type $L^{p(\cdot)}$ norms.

# Decompositions

- As usual, we denote by $\mathrm{X} \oplus_p \mathrm{Y}$ the direct sum of Banach spaces $\mathrm{X}$ and $\mathrm{Y}$ with the norm given by

$$\|(x, y)\|_{\mathrm{X}\oplus_p\mathrm{Y}} = \|x\|_{\mathrm{X}} \boxplus_p \|y\|_{\mathrm{Y}}, \quad x \in \mathrm{X},\ y \in \mathrm{Y},\ 1 \leq p < \infty$$

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$$a \boxplus_p b := (a^p + b^p)^{\frac{1}{p}}, \quad a, b \geq 0.$$

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- Recall that

$$\|f\|_{L^p} = \|1_\Delta f\|_{L^p} \boxplus_p \|1_{[0,1]\setminus\Delta} f\|_{L^p}$$

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- Now, if $p(\cdot)$ is a simple function with values $p_1, p_2, \ldots, p_n$ then it seems reasonable to ask, how and with what isomorphism constants we can decompose $L^{p(\cdot)}$ to the corresponding bands $X_i$. For instance,

$$L^{p(\cdot)} \overset{?}{\approx} (\ldots(X_1 \oplus_{p_2} X_2) \oplus_{p_3} X_3) \oplus_{p_4} \ldots \oplus_{p_n} X_n.$$

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- Preferably, the isomorphism constant should not depend on $n$.

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- But is it easy to analyze these norms locally?
- In the analysis of classical integral operators different $\ell^p$ like auxiliary structures have turned out to be crucial.

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# References on such applications


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# Typical estimates in the paper

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## Theorem

*Let $p \in L^0$, $p \geq 1$ and $\Delta_i$, $i = 1, \ldots, n$, be a measurable decomposition of $[0,1]$, $f \in L^\infty$ and $r_i = \operatorname{ess\,inf}_{\Delta_i} p$, $s_i = \operatorname{ess\,sup}_{\Delta_i} p$. Then*

$$\begin{aligned}\frac{1}{2}(\|\|1_{\Delta_1} f\|\|_{p(\cdot)} \boxplus_{s_2} \|\|1_{\Delta_2} f\|\|_{p(\cdot)}) \boxplus_{s_3} \ldots \boxplus_{s_n} \|\|1_{\Delta_n} f\|\|_{p(\cdot)} \\ \leq \|\|f\|\|_{p(\cdot)} \\ \leq 2(\|\|1_{\Delta_1} f\|\|_{p(\cdot)} \boxplus_{r_2} \|\|1_{\Delta_2} f\|\|_{p(\cdot)}) \boxplus_{r_3} \ldots \boxplus_{r_n} \|\|1_{\Delta_n} f\|\|_{p(\cdot)}.\end{aligned}$$

*and*

$$\begin{aligned}\frac{1}{12}(\|1_{\Delta_1} f\|_{r_1} \boxplus_{r_2} \|1_{\Delta_2} f\|_{r_2}) \boxplus_{r_3} \ldots \boxplus_{r_n} \|1_{\Delta_n} f\|_{r_n} \\ \leq \|\|f\|\|_{p(\cdot)} \\ \leq 12(\|1_{\Delta_1} f\|_{s_1} \boxplus_{s_2} \|1_{\Delta_2} f\|_{s_2}) \boxplus_{s_3} \ldots \boxplus_{s_n} \|1_{\Delta_n} f\|_{s_n}.\end{aligned}$$

# Norms defined by ODEs

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- The absolutely continuous solution is defined by

$$\varphi_f(0) = 0^+,\ \varphi_f'(t) = \frac{|f(t)|^{p(t)}}{p(t)} \varphi(t)^{1-p(t)} \quad \text{for a.e. } t \in [0,1], \qquad (1)$$

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  in the sense of Carathéodory.
- This produces a unique non-negative, non-decreasing solution for a suitable $f \in L^0$, e.g. $f \in L^\infty$.

# Function spaces defined by ODEs

- Given a measurable $p\colon [0,1] \to [1,\infty)$, we eventually define

$$L^{p(\cdot)}_{\mathrm{ODE}} := \{f \in L^0[0,1] \colon \exists\ \varphi_f \ \wedge\ \varphi_f(1) < \infty\}$$

where $\varphi_f$ exists as Carathéodory's weak solution to

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- For instance, if $p(\cdot)$ is essentially bounded then $L^{p(\cdot)}_{\mathrm{ODE}}$ becomes a Banach space.

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# Illustration

- If $p$ is a constant then the function $\varphi_f$ is clearly absolutely continuous and in fact the above ODE is separable.

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$$\varphi(1) = \left(\int_0^1 |f(t)|^p\ dt\right)^{\frac{1}{p}}.$$

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$$L^{p(\cdot)}_{\mathrm{ODE}} = L^{p_1}[0, 1/2] \oplus_{p_2} L^{p_2}[1/2, 1]$$
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- This works similarly for successive $\ell^p$ summations.

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## Proposition

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$$\|f\|_{L^{p(\cdot)}_{\mathrm{ODE}}} \geq \inf\left\{\lambda > 0 \colon \int_0^1 \frac{1}{p(t)} \left(\frac{|f(t)|}{\lambda}\right)^{p(t)} dt \leq 1\right\}$$

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- Then

$$\varphi_f'(t) \geq \frac{|f(t)|^{p(t)}}{p(t)}\lambda^{1-p(t)},$$

so that

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- This is equivalent to

$$\int_0^1 \frac{1}{p(t)}\left(\frac{|f(t)|}{\lambda}\right)^{p(t)} \, dt \leq 1.$$

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- Then

$$\varphi_f'(t) \leq \frac{|f(t)|^{p(t)}}{p(t)}, \quad \text{for a.e. } t \in [t_0, 1].$$

- Thus

$$\varphi_f(1) \leq 1 + \int_{t_0}^1 \frac{|f(t)|^{p(t)}}{p(t)}\, dt \leq 1 + \int_0^1 \frac{|f(t)|^{p(t)}}{p(t)}\, dt = 1 + |||f|||_{p(\cdot)} = 2.$$

Merci !