Intersection properties of the unit ball

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We say that $E \subset X^*$ is 1-*norming* (or "exactly norming") if

$$\|x\| = \sup_{x^* \in E} x^*(x) \quad \text{for each } x \in X;$$

or equivalently (by the Hahn-Banach theorem), $\overline{\text{conv}}^{w^*} E = B_{X^*}$.

Where it started (motivation)

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A similar question with $\|\cdot\|_n \searrow \|\cdot\|$.

Answer 2.
Similarly: NO in general for nonreflexive X, but sometimes YES (e.g., $X = \ell_\infty$).

The properties

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X has property:

- (I) if for every sequence $\{B_n\}_{n\in\mathbb{N}}$ of equivalent balls such that $B_X = \bigcap_n B_n$, one has $B_{X^{**}} = \bigcap_n B_n^{\circ\circ}$;

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- (U) if for every sequence $\{B_n\}_{n\in\mathbb{N}}$ of equivalent balls such that $B_n \overline{\nearrow} B_X$, one has $B_n^{\circ\circ} \overline{\nearrow} B_{X^{**}}$.

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3. *for each 1-norming set $E \subset X^*$, one has* $\text{(I)-env}\,(E) = B_{X^*}$.

Here $\text{(I)-env}\,(E)$ is the *intermediate envelope* of $E \subset X^*$, defined in [*Fonf–Lindenstrauss, Israel J. Math., 2003*] as

$$\text{(I)-env}\,(E) = \bigcap \left\{ \overline{\bigcup_n \overline{\text{conv}}^{w^*}(E_n)} \ : \ E_n \nearrow E \right\}$$

Notice that $\overline{\text{conv}}\, E \subset \text{(I)-env}\,(E) \subset \overline{\text{conv}}^{w^*} E$.

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2. *for each* 1*-norming* $E \subset X^*$*, one has* $\overline{\mathrm{conv}}\, E = B_{X^*}$*;*
3. *for each* $x^{**} \in X^{**} \setminus B_{X^{**}}$*, one has*

$$X \cap \mathrm{conv}\big(B_{X^{**}} \cup \{x^{**}\}\big) \neq B_X \ (= X \cap B_{X^{**}})$$

(this is a kind of maximality of $B_{X^{**}}$*).*

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Moreover,

$$\big[\,(a) \text{ and } (\mathrm{I})\,\big] \;\Rightarrow\; X^* \text{ is separable.}$$

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Corollary

If X satisfies (a) above, then:

$$X \text{ has } (\mathrm{I}) \;\Leftrightarrow\; X \text{ is a separable Asplund space.}$$

Characterizations of reflexivity

Obviously, every reflexive space has (I), (GI), (U).

Theorem

TFAE:

1. X *is reflexive;*
2. X *is dual and has* (GI)*;*
3. *every renorming of* X *has* (I)*;*
4. *every renorming of* X *has* (GI)*;*
5. *every renorming of* X *has* (U).

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Problem (related to 2 above)

If X *is dual and has* (I), *is* X *necessarily reflexive?*

$C(K)$ spaces

Let K be a compact Hausdorff topological space.

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Theorem (Property (GI) in $C(K)$ spaces)

$C(K)$ has (GI) *if and only if K is finite.*

Consequently, if K is metrizable and infinite then $C(K)$ fails (I).

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2. [Sufficient condition.] *If every nonempty G_δ set in K contains a nonempty clopen set then $C(K)$ has* (I)*.*

Consequently, if K is zero-dimensional (that is, clopen sets form a base of the topology) *[e.g., if K is scattered] then $C(K)$ has* (I) *if and only if each nonempty G_δ set in K has nonempty interior.*

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5. $\ell_\infty(\Gamma)^*$ *fails* (I).

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2. *X has the* (MIP)*.*
3. *X is polyhedral.*
4. *The duality mapping $D\colon S_X \to 2^{S_{X^*}}$ is norm-to-weak u.s.c. (E.g., X is* very smooth*.)*

Property (U)

Recall that X *has property* (U) if:
for every sequence $\{B_n\}_{n\in\mathbb{N}}$ of equivalent balls such that $B_n \overline{\nearrow} B_X$,
one has $B_n^{\circ\circ} \overline{\nearrow} B_{X^{**}}$.

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We say that X^* *has property* (I^*) if:
for each sequence $\{D_n\}_n$ of equivalent dual balls such that
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X is a *Grothendieck space* if every w^*-convergent sequence in X^*
is w-convergent.

Characterizations of (U)

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4. (I)-env$(B_X) = B_{X^{**}}$*;*
5. *for each bounded* $\{x_n^*\}_n \subset X^*$ *with* $w^*_{X^*}$*-cluster points contained in* B_{X^*}*, its* $w^*_{X^{***}}$*-cluster points are contained in* $B_{X^{***}}$*;*

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6. X *is "1-Grothendieck" (a quantitative version of the Grothendieck property, introduced and studied by H. Bendová J. Math. Anal. Appl., 2014).*

Spaces having (U)

Theorem

1. *If* X *has* (U) *then* X *is a Grothendieck space.*
2. ℓ_∞ *and* ℓ_∞/c_0 *have* (U).
3. *Property* (U) *passes to quotients, but not to subspaces.*

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Remark (Characterization of (U) in $C(K)$ spaces)

O.F.K. Kalenda [Israel J. Math., 2007] *characterized $C(K)$ spaces such that (I)-env$(B_{C(K)}) = B_{C(K)^{**}}$ (that is, having property (U)) in terms of sequences of Radon probability measures on K.*

The *ball topology* is the smallest (not necessarily Hausdorff) topology in which every closed ball is closed.

Definition

X has the property:

- (BGP) [*"ball generated property"*] if every closed bounded convex set is ball-closed;
- (P) if for each $x^* \in X^*$, the restriction $x^*|_{B_X}$ is ball-sequentially continuous;
- (N) [*"nicely smooth"*] if X^* has no proper subspaces Y such that $B_Y \subset B_{X^*}$ is 1-norming for X.

Analogies (GI)-(BGP) and (I)-(P) – continuation

Theorem (Chen–Hu–Lin, Jimenez Sevilla–Moreno)

If X is Asplund then TFAE:

(a) $X^* = \overline{\text{span}}\,(w^*\text{-strexp}\, B_{X^*})$;

(b) X *has* (BGP);

(c) X *has* (N).

Moreover, for separable X any of the above conditions is equivalent to (P) *and implies that X^* is separable.*

Theorem

If X is Asplund then TFAE:

(a) $B_{X^*} = \overline{\text{conv}}\,(w^*\text{-strexp}\, B_{X^*})$;

(b) X *has* (GI).

Moreover, for separable X any of the above conditions is equivalent to (I) *and implies that X^* is separable.*

Some open problems

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For Γ uncountable, does $\ell_1(\Gamma)$ admit a renorming with (I)?
(For $\Gamma = \mathbb{N}$, no: its dual is not separable!)

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(For $\Gamma = \mathbb{N}$, no: its dual is not separable!)

Problem

If X has (I)*, does it contain a separable subspace with* (I)*?*

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Theorem

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3. X *has* (GI) $\Leftrightarrow \forall$ *(symmetric)* 1-*norming* $A \subset X^*$, $\overline{\text{conv}}\, A = B_{X^*}$.
4. X *has* (U) $\Leftrightarrow X^*$ *has* (I*) $\Leftrightarrow$ (I)-env $(B_X) = B_{X^{**}} \Leftrightarrow X$ *is* 1-*Grothendieck.*

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4. X *has* (U) $\Leftrightarrow X^*$ *has* (I*) $\Leftrightarrow$ (I)-env $(B_X) = B_{X^{**}} \Leftrightarrow X$ *is* 1-*Grothendieck.*

Thank you for attention!