Compatible complex structures on Kalton-Peck space

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Part of this talk is a joint work with professors: J. M. F. Castillo, V. Ferenczi, Y. Moreno.

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Example: $\overline{X^I} = X^{-I}$

Banach spaces without complex structures

- Odd-dimensional Banach spaces.
- James space. J. Dieudonné (1952).
- S. Szarek (1986): There exists a uniformly convex Banach space without complex structure.
- P. Koszmider, M. Martín and J. Merí (2009): Extremely non complex Banach spaces:

$$\|Id + T^2\| = 1 + \|T\|^2$$

- Spaces with few operators: $T = \lambda Id + S$ (Gowers-Maurey space, among others)

Uniqueness of complex structures

Theorem (Kalton, 2009)

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Does any real Banach space with subsymmetric basis admit unique complex structure?

Subsymmetric case

Proposition (C.)

Let E be a real Banach space with a subsymmetric basis $\{x_n\}_n$ such that $E[E] \simeq E$. Then E has unique complex structure.

- For every real Banach space X and J a complex structure on X, $X^J \oplus X^{-J} \simeq X \oplus_{\mathbb{C}} X$.
- P. G Casazza e Bor-Luh Lin (1974) Let X be a Banach space with a subsymmetric basis $\{x_n\}_n^\infty$. Then, for every bounded projection Q on X , either QX or $(Id - Q)X$ contains a subspace isomorphic to X which is complemented in X.
- Let X be a real Banach space with a subsymmetric basis. Then, for every complex structure X^J on X, X^J and $X \oplus_{\mathbb{C}} X$ are complemented one into the other.

Several complex structures

- V. Ferenczi (2007): There exists a real H. I. Banach space $X(\mathbb{C})$ with exactly 2 complex structures.
- $X(\mathbb{C})^n$ has $n+1$ complex structures.
- C. (2014) Space with ω complex structures.
- J. Bourgain (1986) and R. Anisca (2003): examples with $2^{\aleph_0}$ complex structures.

Twisted sums

Definition

Let X and Y be two Banach spaces. A twisted sum of X and Y is a quasi-Banach space Z which contains a subspace $X' \subseteq Z$ isomorphic to X such that the quotient Z/X' is isomorphic to Y.

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Every twisted sum is equivalent to one of the type $X \oplus_\Omega Y$ for a quasi-linear map $\Omega : Y \to X$.

$X \oplus_\Omega Y$ is the space $X \times Y$ endowed with the quasi-norm:

$$\|(x,y)\|_\Omega = \|x - \Omega y\| + \|y\|$$

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Z_2 Kalton-Peck space

$Z_2 = \ell_2 \oplus_{\Omega_2} \ell_2$ is the twisted Hilbert space obtained by considering the non-trivial quasi-linear map (defined on finitely supported sequences)

$$\Omega_2(x)(n) = x(n) \log \frac{\|x\|}{|x(n)|}.$$

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Conjecture

Z_2 is not isomorphic to its hyperplanes.

Complex structures on Z_2 and its hyperplanes

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Hiperplanes of Z_2 do not admit complex structure.

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Hiperplanes of Z_2 do not admit complex structure.

Theorem

No complex structure on ℓ_2 can be extended to a complex structure on the hyperplane $\ell_2 \oplus_{\Omega_2 i} H$.

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No complex structure on ℓ_2 can be extended to a complex structure on the hyperplane $\ell_2 \oplus_{\Omega_2 i} H$.

Let I be a complex structure on a real Banach space X and H be an hyperplane of X. Then $I|_H$ is not a complex structure on H.

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Theorem (V. Ferenczi, E. Galego, 2007)

Let T, u be complex structures on, respectively, an infinite dimensional Banach space X and some hyperplane H of X. Then the operator $T|_H - u$ is not strictly singular.

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Corollary

Let α be any complex structure on ℓ_2 and let $\ell_2 \oplus_{\Omega_2 i} H$ be a canonical hyperplane of Z_2. Then α can not be extended to a complex structure on Z_2 and $\ell_2 \oplus_{\Omega_2 i} H$, simultaneously.

Non compatible complex structure

Question

Does every complex structure on ℓ_2 admit an extension to a complex structure on Z_2?

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Theorem

There exists a complex structure U on ℓ_2 that can not be extended to any operator on Z_2.

Non compatible complex structure

Let $\mathcal{B}$ be an orthonormal basis of ℓ_2 containing the vectors

$$\{f_k^n, e_{2^{n+1}+k} \,:\, n \text{ even},\ 1 \leq k \leq n\},$$

$$
\begin{aligned}
f_1^n &= \frac{1}{\sqrt{2^n}} \sum_{j=1}^{2^n} (-1)^{j+1} e_{2^n+j-1} \\
f_2^n &= \frac{1}{\sqrt{2^n}} \sum_{j=1}^{2^{n-1}} (-1)^{j+1} (e_{2^n+2(j-1)} + e_{2^n+2j-1}) \\
&\vdots \\
f_n^n &= \frac{1}{\sqrt{2^n}} \left((e_{2^n} + \cdots + e_{2^n+2^{n-1}+1}) - (e_{2^n+2^{n-1}} + \cdots + e_{2^{n+1}-1}) \right)
\end{aligned}
$$

Non compatible complex structure

Define a complex structure U on ℓ_2 by setting

$$U(f_k^n) = e_{2^{n+1}+k} \ \ (k = 1, \dots, n)$$

$$U(e_{2^{n+1}+k}) = -f_k^n \ \ (k = 1, \dots, n)$$

when n is an even number (and extending properly in all ℓ_2).
For all n (even) large enough:

$$Ave_{\pm} \| [U, \Omega_2] (\sum_{k=1}^{n} \epsilon_k f_k^n) - \sum_{k=1}^{n} \epsilon_k [U, \Omega_2] f_k^n \| \geq (1/10) \sqrt{n} \log n.$$

This implies that U can not be extended to any operator on Ω_2.

Compatible complex structures on Z_2 and its hyperplanes

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No complex structure on ℓ_2 can be extended to a complex structure on the hyperplane $\ell_2 \oplus_{\Omega_2 i} H$.

Proof: Let u be a complex structure on ℓ_2. Suppose that can be extended to U on $\ell_2 \oplus_{\Omega_2 i} H$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ell_2 & \longrightarrow & \ell_2 \oplus_{\Omega_2 i} H & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u} & & \vdots{\scriptstyle U} & & \vdots{\scriptstyle v} & & \\
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Extending v to a complex structure V on ℓ_2, we have that (u, V) is compatible with Ω_2.

Compatible complex structures on Z_2 and its hyperplanes

Definition

The pair of operators (u, V) is said to be compatible with Ω_2

$$\begin{array}{ccccccccc}
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Compatible complex structures on Z_2 and its hyperplanes

Definition

The pair of operators (u, V) is said to be compatible with Ω_2 if there exists an operator U such that the diagram is commutative.

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Proposition. The pair (u, V) is compatible with Ω_2 iff $u\Omega_2 - \Omega_2 V$ is trivial.

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Proposition. The pair (u, V) is compatible with Ω_2 iff $u\Omega_2 - \Omega_2 V$ is trivial.

We prove that $u - V$ is compact, thus obtaining a contradiction with Ferenczi-Galego theorem.

Compatible complex structures on Z_2 and its hyperplanes

Proposition

For every operator $T : \ell_2 \to \ell_2$, and for every block subspace W of ℓ_2, the commutator $\Omega_2 T - T\Omega_2$ is trivial on some block subspace of W.

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Proof u-V is compact.

Suppose by contradiction that $(u - V)|_W$ is an isomorphism for some block subspace W of ℓ_2.

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For every operator $T : \ell_2 \to \ell_2$*, and for every block subspace* W *of* ℓ_2*, the commutator* $\Omega_2 T - T\Omega_2$ *is trivial on some block subspace of* W*.*

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$$
\begin{aligned}
(u - V)\Omega_2 &= u\Omega_2 - \Omega_2 V \\
&- (\Omega_2 V - V\Omega_2)
\end{aligned}
$$

Then Ω_2 is trivial on $(u - V)^{-1}W'$. A contradiction with the fact that Ω_2 is singular.

□

Questions

- Do hyperplanes of Z_2 admit complex structure?
- If K is a compact operator on ℓ_2, must both $\Omega_2 K$ and $K\Omega_2$ be trivial?
- Let $\ell_2 \oplus_\Omega \ell_2$ be a *singular* twisted sum, and (T, U) a pair of compatible operators with Ω. Then, is $T - U$ compact?

Thanks for the attention!

Construction: Non compatible complex structure

Proposition

Let G be trivial on ℓ_2. Then for every $x_1, x_2, \ldots, x_n$ in ℓ_2,

$$Ave_{\pm} \left\| G\left(\sum_{k=1}^{n} \epsilon_k x_k\right) - \sum_{k=1}^{n} \epsilon_k G(x_k) \right\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2}.$$

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where $f_k^n \in span\{e_{2^n}, \ldots, e_{2^{n+1}-1}\}$,

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Let $\mathcal{B}$ be an orthonormal basis of ℓ_2 containing the vectors

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where $f_k^n \in span\{e_{2^n}, \ldots, e_{2^{n+1}-1}\}$, and for every sequence of signs $(\epsilon_1, \ldots, \epsilon_n)$ there exists (unique) $l \in \{2^n, \ldots, 2^{n+1} - 1\}$ such that

$$(\epsilon_1, \ldots, \epsilon_n) = \sqrt{2^n}(f_1^n(l), \ldots, f_n^n(l))$$

Construction: Non compatible complex structure

Proposition

Let G be trivial on ℓ_2. Then for every $x_1, x_2, \ldots, x_n$ in ℓ_2,

$$Ave_{\pm} \left\| G\left(\sum_{k=1}^{n} \epsilon_k x_k\right) - \sum_{k=1}^{n} \epsilon_k G(x_k) \right\| \leq C \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{1/2}.$$

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$$\{f_k^n, e_{2^{n+1}+k} : n \text{ even}, 1 \leq k \leq n\},$$

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Construction: Non compatible complex structure

Define a complex structure U on ℓ_2 by setting

$$U(f_k^n) = e_{2^{n+1}+k} \ \ (k = 1, \dots, n)$$

$$U(e_{2^{n+1}+k}) = -f_k^n \ \ (k = 1, \dots, n)$$

when n is an even number (and extending properly in all ℓ_2).

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when n is an even number (and extending properly in all ℓ_2).

Proposition

For all n (even) large enough:

$$Ave_{\pm}\|[U, \Omega_2](\sum_{k=1}^{n} \epsilon_k f_k^n) - \sum_{k=1}^{n} \epsilon_k [U, \Omega_2] f_k^n\| \geq (1/10)\sqrt{n}\log n.$$

In particular, (U, U) is not compatible with Ω_2.

Construction: Non compatible complex structure

(T, U) is not compatible with Ω_2

Let T be any operator on ℓ_2, then $[T, \Omega_2, U]$ is not trivial. In fact

$$Ave_{\pm}\|[T, \Omega_2, U](\sum_{k=1}^{n} \epsilon_k f_k^n) - \sum_{k=1}^{n} \epsilon_k [T, \Omega_2, U] f_k^n\| \geq (1/20)\sqrt{n}\log n$$

(U, T) is not compatible with Ω_2

(U, T) compatible with $\Omega_2 \Rightarrow (T^*, U^*)$ compatible with $\Omega_2^* \Rightarrow (-T^t, U)$ compatible with Ω_2.

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