Dunford-Pettis-type operators as Ideal Extensions

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Outline of talk

1. Introduce and discuss extensions of operator ideals.
2. Introduce the classes of p-convergent and weak* p-convergent operators.
3. Discuss their properties as the extensions of the ideals of classes of operators.

Definitions and notation from literature I

- The space of all weakly p-summable sequences in a Banach space X is denoted by $\ell_p^{weak}(X)$, $1 \leq p < \infty$;
- Recall that it is a Banach space with norm

$$\|(x_i)\|_p^{weak} := \sup \left\{ \left(\sum_{i=1}^{\infty} |\langle x_i, x^* \rangle|^p \right)^{1/p} : x^* \in X^*, \|x^*\| \leq 1 \right\}.$$

- This space is isometrically isomorphic to $\mathcal{L}(\ell_{p'}, X)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$).
- In the case of $p = \infty$ we consider the space $c_0^{weak}(X)$ of weakly null sequences in X.

Definitions and notation from literature II

- The closed subspace

$$\ell_p^u(X) = \{(x_n) : (x_n) = \lim_{n\to\infty}(x_1, \cdots x_n, 0, \cdots)\},$$

where $\ell_p^u(X) \subset \ell_p^{weak}(X)$, is isometrically isomorphic to the space of compact operators $\mathcal{K}(\ell_{p'}, X)$.

- Fourie and Aywa showed that $(x_n) \in \ell_p^u(X)$ if, and only if, $(x_n) \in \ell_p^{weak}(X)$ and $\|x_n\| \xrightarrow[\infty]{n} 0$.
- The space of all weak* p-summable sequences in the dual space X^* of a Banach space X is denoted by $\ell_p^{weak^*}(X^*)$, $1 \leq p < \infty$.

Definitions and notation from literature III

- Recall that it is a Banach space with norm

$$\|(x_i^*)\|_p^{weak^*} := \sup\left\{\left(\sum_{i=1}^{\infty}|\langle x, x_i^*\rangle|^p\right)^{1/p} : x \in X, \|x\| \leq 1\right\}.$$

- This space is isometrically isomorphic to $\mathcal{L}(X, \ell_p)$.
- In the case of $p = \infty$ we consider the space $c_0^{weak^*}(X^*)$ of weak* null sequences in X^* and replace ℓ_∞ by c_0.
- The closed subspace

$$\ell_p^u(X^*) = \{(x_n^*) : (x_n^*) = \lim_{n\to\infty}(x_1^*, \cdots x_n^*, 0, \cdots)\},$$

where $\ell_p^u(X^*) \subset \ell_p^{weak^*}(X^*)$, is isometrically isomorphic to the space of compact operators $\mathcal{K}(X, \ell_p)$.

Definition

Let *BAN* be the family of all Banach spaces. Consider a Banach operator ideal $(\mathcal{A}, \alpha)$. Recall that this implies in particular that for all Banach spaces X, Y the *components* $(\mathcal{A}(X, Y), \alpha(\cdot))$ are complete normed linear subspaces of the space $\mathcal{L}(X, Y)$ of bounded linear operators that satisfy:

1. $x^* \otimes y : X \to Y : x \mapsto x^*(x)y$ belongs to $\mathcal{A}(X, Y)$, for all $x^* \in X^*$ and $y \in Y$ and
$$\alpha(x^* \otimes y) = \|x^*\|\|y\|.$$
2. If $T \in \mathcal{L}(X_0, X)$, $S \in \mathcal{A}(X, Y)$ and $R \in \mathcal{L}(Y, Y_0)$, then $RST \in \mathcal{A}(X_0, Y_0)$ and
$$\alpha(RST) \leq \|R\|\alpha(S)\|T\|.$$
3. $\|T\| \leq \alpha(T)$ for all $T \in \mathcal{A}(X, Y)$.

Fix a Banach (scalar) sequence space $(\Lambda, \|\cdot\|_\Lambda)$ which contains the set ϕ of all sequences having only a finite number of non-zero terms and for which the set $\{e_n : n \in \mathbb{N}\}$ satisfies $\|e_n\| = 1$ for all n. For instance, when $\Lambda = \ell_p$ (for $1 \leq p \leq \infty$) or $\Lambda = c_0$.

Definition

For $X, Y \in BAN$, let

$$\mathcal{A}_\Lambda(X, Y) := \{T \in \mathcal{L}(X, Y) : ST \in \mathcal{A}(X, \Lambda), \forall S \in \mathcal{L}(Y, \Lambda)\}.$$

Clearly,
$\mathcal{A}(X, Y) \subset \mathcal{A}_\Lambda(X, Y)$ and $\mathcal{A}_\Lambda(X, \Lambda) = \mathcal{A}(X, \Lambda)$ for all Banach spaces X, Y.

Moreover,

Theorem

$(\mathcal{A}_\Lambda, \alpha_\Lambda)$ *is a Banach operator ideal with ideal norm defined by*

$$\alpha_\Lambda(T) = \sup\{\alpha(ST) : S \in \mathcal{L}(Y, \Lambda), \|S\| \leq 1\}$$

for all $T \in \mathcal{A}_\Lambda(X, Y)$ *and all Banach spaces* X *and* Y*.*

We call the pair $(\mathcal{A}_\Lambda, \alpha_\Lambda)$ the "right"-Λ-extension of the ideal $(\mathcal{A}, \alpha)$.

Similarly,

Definition

For a Banach operator ideal $(\mathcal{A}, \alpha)$, we consider the Banach operator ideal $(\mathcal{A}_\Lambda^\diamondsuit, \alpha_\Lambda^\diamondsuit)$, where $\mathcal{A}_\Lambda^\diamondsuit(X, Y) =$

$$\{T \in \mathcal{L}(X, Y) : TS \in \mathcal{A}(\Lambda, Y), \forall S \in \mathcal{L}(\Lambda, X)\}$$

and

$$\alpha_\Lambda^\diamondsuit(T) = \sup\{\alpha(TS) : S \in \mathcal{L}(\Lambda, X), \|S\| \leq 1\}.$$

We call the pair $(\mathcal{A}_\Lambda^\diamondsuit, \alpha_\Lambda^\diamondsuit)$ the "left"-Λ-extension of the ideal $(\mathcal{A}, \alpha)$.

Remarks

1. Henceforth, we assume in addition that the scalar sequence space Λ is a normal BK-space with AK.
2. This implies that if $(\alpha_i) \in \Lambda$ and $|\lambda_i| \leq |\alpha_i|$ for all i, then $(\lambda_i) \in \Lambda$, the coordinate projections $(\lambda_i) \mapsto \lambda_j$ are continuous for all j and that $\{e_n : n \in \mathbb{N}\}$ is a Schauder basis for Λ (the AK-property).
3. Moreover, if the dual space Λ^* of the normal BK-space Λ with AK is again a normal BK-space with the AK-property, then we call Λ a DAK-space.
4. For instance, if Λ is a reflexive normal BK-space with AK, then it is a DAK-space.

p-convergent operators

Let $1 \leq p < \infty$. We denote the family of p-convergent operators (introduced by J.M.F. Castillo and F. Sanchez) on BAN by $\mathcal{C}_p$.
An operator $T \in \mathcal{L}(X, Y)$ belongs to $\mathcal{C}_p(X, Y)$ if for each weakly p-summable sequence (x_n) in X, $\lim_{n\to\infty} \|Tx_n\| = 0$.

Proposition

$$T \in \mathcal{C}_p(X, Y) \iff TS \in \mathcal{K}(\ell_{p'}, Y), \forall S \in \mathcal{L}(\ell_{p'}, X).$$

Thus,

$$\mathcal{K}^{\diamondsuit}_{\ell_{p'}}(X, Y) = \mathcal{C}_p(X, Y),$$

for all Banach spaces X, Y.

Proof I

($\Rightarrow$) Let $T \in \mathcal{C}_p(X,Y)$ and $S \in \mathcal{L}(\ell_{p'}, X)$.

Then $S((\lambda_n)) = \sum_n \lambda_n x_n$ for a unique $(x_n) \in \ell_p^{weak}(X)$ and $\|Tx_n\| \xrightarrow[\infty]{n} 0$.

Therefore $(Tx_n) \in \ell_p^u(Y)$ and thus $TS((\lambda_n)) = \sum_n \lambda_n Tx_n$, i.e. $TS \in \mathcal{K}(\ell_{p'}, Y)$ for each $S \in \mathcal{L}(\ell_{p'}, X)$.

This shows that $T \in \mathcal{K}_{\ell_{p'}}^{\diamondsuit}(X,Y)$.

($\Leftarrow$) Conversely, let $T \in \mathcal{K}_{\ell_{p'}}^{\diamondsuit}(X,Y)$.

This implies that $TS \in \mathcal{K}(\ell_{p'}, Y), \forall S \in \mathcal{L}(\ell_{p'}, X)$.

Therefore, $(Tx_n) \in \ell_p^u(Y), \forall (x_n) \in \ell_p^{weak}(X)$.

Hence, $\|Tx_n\| \xrightarrow[\infty]{n} 0, \forall (x_n) \in \ell_p^{weak}(X)$.

This shows that $T \in \mathcal{C}_p(X,Y)$. □

Remark

This shows that $\mathcal{C}_p(X, Y)$ is the "left"– $\ell_{p'}$– extension of the Banach operator ideal $\mathcal{K}(X, Y)$.

Ideal of p-convergent operators

Although no norm has been considered on $\mathcal{C}_p(X, Y)$ before, it now follows that if we let

$$c_p(T) := \|T\|^{\diamondsuit}_{\ell_{p'}} = \sup_{\substack{\|S\|\leq 1 \\ S\in\mathcal{L}(\ell_{p'},X)}} \|TS\|,$$

then $(\mathcal{C}_p, c_p)$ is a Banach operator ideal.

Weak* p-convergent operators

Zeekoei and Fourie introduced the so-called weak* p-convergent operators.

Definition

An operator T from a Banach space X into a Banach space Y is said to be weak* p-convergent if $(y_n^*(Tx_n))$ converges to 0 for every $(x_n) \in \ell_p^{weak}(X)$ and every $(y_n^*) \in c_0^{weak^*}(Y^*)$.

It follows easily from the definition that:

Theorem

If $S : X \to Y$ is weak* p-convergent, $T \in \mathcal{L}(X_0, X)$, $R \in \mathcal{L}(Y, Y_0)$, then $RST : X_0 \to Y_0$ is weak* p-convergent.

In the following theorem the connection between the weak* p-convergent and p-convergent operators is discussed:

Theorem

Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. The following are equivalent:

(a) *T is weak* p-convergent.*

(b) *ST is p-convergent for each $S \in \mathcal{L}(Y, Z)$ and any separable Banach space Z.*

(c) *ST is p-convergent for each $S \in \mathcal{L}(Y, c_0)$.*

Proof I

Clearly, $(b) \implies (c)$.

We prove: $(a) \implies (b)$:

Assume T is weak* p-convergent.

Suppose ST is not p-convergent for some $S \in \mathcal{L}(Y, Z)$ and some separable Banach space Z.

Then there exists $(x_n) \in \ell_p^{weak}(X)$ such that $\|STx_n\| \nrightarrow 0$ as $n \to \infty$.

Taking subsequences we may therefore assume that there exist an $\varepsilon > 0$ and a sequence $(y_n^*) \subset \mathcal{B}_{Z^*}$ such that $|\langle y_n^*, STx_n \rangle| \geq \varepsilon$ for all $n \in \mathbb{N}$.

Since Z is separable and $\mathcal{B}_{Z^*}$ is weak*-compact, there exists a subsequence $(y_{n_k}^*)$ such that $y_{n_k}^* \to y^* \in \mathcal{B}_{Z^*}$ weak*.

Proof II

Since $S^* : Z^* \to Y^*$ is weak*-weak* continuous, we have $S^* y^*_{n_k} \to S^* y^*$ weak*.

Put $h^*_{n_k} = S^* y^*_{n_k} - S^* y^*$, i.e. $h^*_{n_k} \to 0$ weak* in Y^*.

By assumption, $\langle h^*_{n_k}, Tx_{n_k} \rangle \xrightarrow[\infty]{k} 0$.

Also, since $STx_{n_k} \to 0$ weakly, because ST is weak-weak continuous, we have

$$\langle y^*_{n_k}, STx_{n_k} \rangle = \langle h^*_{n_k}, Tx_{n_k} \rangle + \langle y^*, STx_{n_k} \rangle \xrightarrow[\infty]{k} 0.$$

This contradicts the assumption $|\langle y^*_{n_k}, STx_{n_k} \rangle| \geq \varepsilon$ for all $k \in \mathbb{N}$.

$(c) \implies (a)$:

Proof III

Suppose for each bounded linear operator S from Y into c_0, the operator ST is p-convergent.

Let $(x_n) \in \ell_p^{weak}(X)$ and $(y_n^*) \in c_0^{weak^*}(Y^*)$.

If we put $S_{(y_i^*)}(y) = (\langle y, y_i^* \rangle)_i$, then $S \in \mathcal{L}(Y, c_0)$ and by assumption ST is p-convergent.

Thus, since $(e_n) \subset c_0^* = \ell_1$ is weak* convergent to 0, it follows that $\langle e_n, STx_n \rangle \xrightarrow[\infty]{n} 0$.

However,

$$\begin{aligned} \langle e_n, STx_n \rangle &= \langle e_n, S(Tx_n) \rangle \\ &= \langle e_n, (\langle Tx_n, y_i^* \rangle)_i \rangle \\ &= \langle y_n^*, Tx_n \rangle. \end{aligned}$$

This proves that T is weak* p-convergent.

Remark

1. If we denote the family of weak* p-convergent operators on BAN by $\mathcal{W}_p^*(X,Y)$, then it follows from the preceding theorem ((a) $\iff$ (c)), and with $\Lambda = c_0$ in Definition 1 that
$$\mathcal{W}_p^*(X,Y) = (\mathcal{C}_p)_{c_0}(X,Y).$$
2. This shows that $\mathcal{W}_p^*(X,Y)$ is the "right"– c_0– extension of the operator ideal $\mathcal{C}_p(X,Y)$.
3. Although we don't have an ideal norm defined on $\mathcal{W}_p^*$, the equality in [1.] suggests that we may put $w_p^*(T) = \|T\|_{(\mathcal{C}_p)_{c_0}}$.
4. Then $(\mathcal{W}_p^*, w_p^*)$ is a Banach operator ideal.

References I

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Thank you for your attendance and attention.