Variational Principle for Integral Functional

Milen Ivanov Nadia Zlateva[1]

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Variational problem

Consider the optimization problem

$$\int_0^\infty \left(\|u'(t)\|^2 + f(u(t))\right)\,dt \to \min$$

where f is closed, convex and such that $f \geq 0$, $f(0) = 0$ with initial condition

$$u(0) = a.$$

This is a simple problem of Calculus of Variations.
Here X is Banach space, $f : X \to \mathbb{R}$ and $u : \mathbb{R} \to X$. If X is reflexive, the problem has a solution.

What if X is not reflexive?

Existence result

Theorem

Let $(X, \|\cdot\|)$ be a Banach space. Let $f : X \to \mathbb{R} \cup \{\infty\}$ be closed, convex and such that $f \geq 0$, $f(0) = 0$ and $f(\cdot) \geq k\|\cdot\|$ for some $k > 0$. Let $a \in \operatorname{dom} f \setminus \{0\}$ be fixed.
Consider the optimization problem

$$(V_{\|\cdot\|}) \begin{cases} \displaystyle\int_0^{\infty} \left(\|v(t)\|^2 + f(u(t))\right) \, dt \to \min \\ u(t) = a + \displaystyle\int_0^t v(s)\, ds, \qquad v \in L^2\left([0,\infty), X\right). \end{cases}$$

For each $\varepsilon > 0$ there is an equivalent norm $|\cdot|$ on X such that

$$\|\cdot\| \leq |\cdot| \leq (1+\varepsilon)\|\cdot\|$$

and the problem $(V_{|\cdot|})$ has a solution.

For $f : X \to \mathbb{R} \cup \{\infty\}$ and $a \in \operatorname{dom} f$ satisfying the conditions of the Theorem we define

$$F_{\|\cdot\|} : L^2([0,\infty), X) \to \mathbb{R} \cup \{\infty\}$$

by

$$F_{\|\cdot\|}(v) = \int_0^\infty \left(\|v(t)\|^2 + f(u(t))\right) dt, \quad u(t) = a + \int_0^t v(s)\, ds.$$

Then the problem $(V_{\|\cdot\|})$ may be reformulated simply as

$$(F_{\|\cdot\|}, Y) \begin{cases} F_{\|\cdot\|}(v) \to \min; \\ v \in Y = L^2([0,\infty), X). \end{cases}$$

If we apply Ekeland variational principle

For $\varepsilon > 0$ we can get $v_\varepsilon \in L^2([0,\infty), X)$ such that

$$F_{\|\cdot\|}(v) + \varepsilon \|v - v_\varepsilon\|_{L^2}$$

attains minimum at v_ε. But

$$\|v - v_\varepsilon\|_{L^2} = \left(\int_0^\infty \|v(t) - v_\varepsilon(t)\|^2 \, dt \right)^{\frac{1}{2}},$$

so the perturbed problem is not alike the initial problem $(V_{\|\cdot\|})$.

Aim

To perturb the function $F_{\|\cdot\|}$ in the problem

$$(F_{\|\cdot\|}, Y) \begin{cases} F_{\|\cdot\|}(v) \to \min; \\ v \in Y \end{cases}$$

in a way that the perturbed function to be of the same type as the former, i.e. to be a function $F_{|\cdot|}$ for some equivalent norm $|\cdot|$ and the perturbed problem

$$(F_{|\cdot|}, Y) \begin{cases} F_{|\cdot|}(v) \to \min; \\ v \in Y \end{cases}$$

to have a solution.

Abstract set up

Here below S is a closed convex subset of the Banach space X and $f : S \to \mathbb{R} \cup \{\infty\}$ is closed, convex, proper and bounded below function. Consider the optimization problem

$$(f, S) \begin{cases} f(x) \to \min; \\ x \in S, \end{cases}$$

and the set of ε-minima of f on S

$$\varepsilon\text{-}\mathrm{argmin}_S f := \{x \in S; f(x) \le \inf_S f + \varepsilon\},$$

Definition

We say that the problem (f, S) is weakly well-posed if

$$\lim_{\varepsilon \downarrow 0} \beta(\varepsilon\text{-}\mathrm{argmin}_S f) = 0,$$

where β is the following measure of weak non-compactness.

Cantor-Kuratowski-De Blasi Lemma

Given a set $A \subset X$, the measure of weak non-compactness of A is

$\beta(A) := \inf\{\varepsilon > 0;$ there is weakly compact B s.t. $A \subset B + \varepsilon B_X\}$.

Lemma

Let $\{A_n\}_{n \in \mathbb{N}}$ be a nested ($A_{n+1} \subset A_n$, $\forall n$) family of weakly closed subsets of a Banach space such that

$$\lim_{n \to \infty} \beta(A_n) = 0.$$

Then $A = \cap A_n$ is nonempty.

Since both S and f are assumed convex and closed, then if the problem (f, S) is weakly well-posed, it has a solution (that is, $\operatorname{argmin}_S f \neq \emptyset$).

Perturbation space relative to (f, S). Axioms

A complete metric space (P, ρ) of positive convex continuous functions from X to $\mathbb{R}$ is perturbation space relative to (f, S) if:
(i) P is a convex cone, that is, $g_i \in P$, $i = 1, 2 \Rightarrow g_1 + g_2 \in P$, and $\forall g \in P,\ c \geq 0 \Rightarrow cg \in P$;
(ii) if $g, g_k \in P$ and $\rho(g_k, 0) \to 0$ then $\rho(g, g + g_k) \to 0$;
(iii) ρ-convergence implies uniform convergence on bounded sets;
(iv) for any $\varepsilon > 0$ there is $t_\varepsilon > 0$ such that:
for any $x \in \operatorname{dom} f \cap S$ and any $\delta > 0$ there is $y \in \operatorname{dom} f \cap S$ such that

$$\|y - x\| < \delta, \quad |f(x) - f(y)| < \delta,$$

and there is $g \in P$ such that $\rho(g, 0) < \varepsilon$, $g(y) < \delta$ and

$$\beta\left(t_\varepsilon\text{-}\mathrm{argmin}_S\, g\right) < \varepsilon.$$

Variational principle

Theorem

Let S be a closed convex and bounded subset of a Banach space X. Let $f : S \to \mathbb{R} \cup \{+\infty\}$ be closed, convex, proper and bounded below.
If (P, ρ) is a perturbation space relative to (f, S), then there exists a dense G_δ subset of $G \subset P$ and $g \in G$ such that $(f + g, S)$ is weakly well-posed.

We can drop the boundedness assumption on S if we assume that in the perturbation space (P, ρ) relative to (f, S) there are $g_k \in P$ with $\rho(g_k, 0) \to 0$ and

$$\lim_{\|x\| \to \infty} g_k(x) = \infty, \quad \forall k \in \mathbb{N}.$$

In our case

The problem

$$(V_{\|\cdot\|})\begin{cases}\displaystyle\int_0^\infty \left(\|v(t)\|^2 + f(u(t))\right)\,dt \to \min \\ \displaystyle u(t) = a + \int_0^t v(s)\,ds, \qquad\qquad v \in L^2\left([0,\infty), X\right).\end{cases}$$

is reformulated as

$$(F_{\|\cdot\|}, Y)\begin{cases}F_{\|\cdot\|}(v) \to \min; \\ v \in Y,\end{cases}$$

where

$$F_{\|\cdot\|}(v) := \int_0^\infty \left(\|v(t)\|^2 + f(u(t))\right)\,dt, \quad u(t) = a + \int_0^t v(s)\,ds$$

is proper, closed, convex and positive on $Y := L^2([0,\infty), X)$.

Perturbation space relative to $\left(F_{\|\cdot\|}, Y\right)$

The cone P over $Y := L^2([0,\infty), X)$ consists of all functions g of the form

$$v \to \int_0^\infty |v(t)|^2 \, dt,$$

where $|\cdot|$ is some equivalent norm on X.
We equip P with a metric ρ in the following way: for two functions $g_i(v) = \int_0^\infty |v(t)|_i^2 \, dt \in P$, $i = 1, 2$,

$$\rho(g_1, g_2) := \sum_{n=1}^\infty 2^{-n} \sup\left\{ \left| \, |x|_1^2 - |x|_2^2 \, \right| ; x \in nB_X \right\}.$$

The so defined (P, ρ) is a perturbation space relative to $(F_{\|\cdot\|}, Y)$.

Why it works?

The stepwise functions are graphically dense in $\operatorname{dom} F_{\|\cdot\|}$, that is, for any $v \in \operatorname{dom} F_{\|\cdot\|}$ there exists a sequence of stepwise w_k such that $w_k \to v$ and $F_{\|\cdot\|}(w_k) \to F_{\|\cdot\|}(v)$.

Let w_1 be a stepwise function. Let $X_1 = \operatorname{span} w_1([0, \infty))$, so $\dim X_1 < \infty$. For appropriate $r > 0$, $r = r(\delta)$, define

$$|x|_1^2 := d^2(x, X_1) + r\|x\|^2,$$

and thus get an equivalent norm $|\cdot|_1$ on X.

Let $g_1 \in P$ be the function $g_1(v) = \int_0^\infty |v|_1^2 \, dt$. Then

$$g_1(w_1) = r\|w_1\|_2^2 < \delta.$$

Set $g(\cdot) := (\varepsilon/M)g_1(\cdot)$.
Then $g(w_1) < \delta$ and $\rho(g, 0) < \varepsilon$ for appropriate choice of M.

Consider the ball $A = r^{-1}B_{Y_1}$, where $Y_1 = L^2([0,\infty), X_1)$. Note that Y_1 is reflexive and therefore A is weakly compact and use this to get the estimate

$$\beta(t_\varepsilon\text{-argmin}_S \, g) < \varepsilon.$$

The same idea works in different situations. Example

Theorem

Let X be a Banach space and $f : X \times [0,1] \to \mathbb{R} \cup \{\infty\}$ be closed, bounded below and such that $\int_0^1 f(0,t)\,dt < \infty$.
Consider the problem

$$(P_f) \begin{cases} \displaystyle\int_0^1 f(\mathbf{u}(t),t)\,dt \to \min \\ \mathbf{u} : [0,1] \to X \text{ is 1-Lipschitz and s.t. } \mathbf{u}(0) = 0. \end{cases}$$

For any $\delta > 0$ there exists positive convex and continuous $g : X \to \mathbb{R}$ with $\sup g(B_X) < \delta$, such that the problem

$$(P_{f+g}) \begin{cases} \displaystyle\int_0^1 f(\mathbf{u}(t),t) + g(\mathbf{u}(t))\,dt \to \min \\ \mathbf{u} : [0,1] \to X \text{ is 1-Lipschitz and s.t. } \mathbf{u}(0) = 0. \end{cases}$$

has a solution.

Publications

M. Ivanov and *N. Zlateva*, J. Convex Anal. 19, No. 4, 1033–1042, 2012.

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THANK YOU FOR THE ATTENTION